THE MATHEMATICAL GAZETTE

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THE GEOMETRY OF MEGALITHIC MAN

BY A. THOM

From one end of Britain to the other Megalithic Man has left us examples of geometric constructions of various designs. Originally set out on the ground by means of stakes, rope and measuring rods, the designs were rendered permanent by large stones placed along the outlines. The erection of the larger of these stones was an enormous task. It is certain that the engineers controlling the operations were working to pre-conceived plans which were set out with care and accuracy. Further, it appears that the unit of length used was the same from Land's End to John o' Groats. That this unit was 5·44 ft. I have shown in Ref. 1, and many more sites surveyed since writing that paper have amply confirmed statistically this value. Half of this, 2·72 ft., might be called the Megalithic Yard, but whether this or the Megalithic Fathom (5·44 ft.) was the length of the standard rod carried about the country it is not possible to say. The important thing is that in a statistically significant number of cases in any group of circles the major diameter is approximately an integral number of fathoms. The group can be formed according to locality, type of circle, or size; the result is the same. We also find this dimension occurring in the distances of outliers from the construction centre and the distances between circles where there are two close together. The method of analysis used was that given by Broadbent in Ref. 5.

Of the 160 circles of which I have made accurate surveys the majority are circular but 35 are definitely flattened at one side. This flattening was evidently intentional and it is proposed to show that it was always done to a geometrical pattern. Two types of pattern appear, type A and type B (Fig. 1). These were probably set out with rope and stakes. We can set a stake at the centre and

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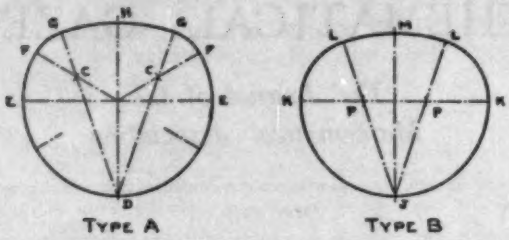
H
G
G
F
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C
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D
TYPE A
L
M
L
K
P
P
K
J
TYPE B

FIG. 1

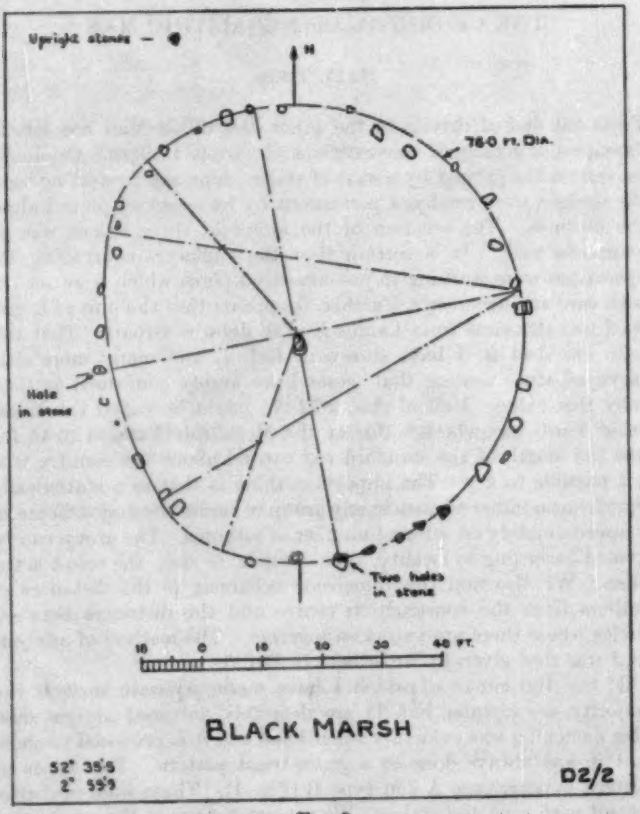
Upright stones
N
76·0 ft. Dia.
Hole in stone
Two holes in stone
10 0 10 20 30 40 FT.
BLACK MARSH
52° 35'5
2° 59'9
D2/2

FIG. 3

scribe a circle round it with a length of rope and a second stake. Evidently the constructors used the property of the chord equal to the radius going round the circle six times. In both type A and type B the circular part of the construction is a multiple of 60°, in type A, 240°, and in type B, 180°. The rope used for scribing could be folded so as to bisect or trisect its length. In type A the radius was thus bisected at the point *C* and the short arc *FG* drawn in. With centre *D* the arc *GG* could now be drawn to complete the periphery. Type B is simpler: the diameter is divided into three equal parts and the points of sub-division *P* and *P* used for the minor arcs. The periphery is then closed as before with an arc centred at *J* on the far side.

It will be seen that both the above constructions can be carried

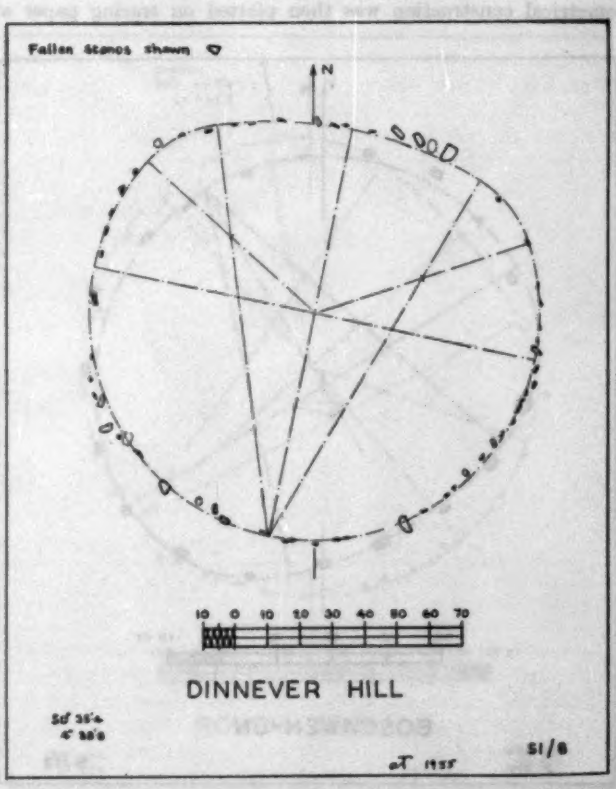

FIG. 2

out on the ground but it is interesting that the constructors realised the necessity of making the arcs of different radius meet on the line joining the centres. In Ref. 1 it is shown that in some cases the points where the radius changes are marked by stones as are the principal axes. At Castle Rigg near Keswick (Ref. 1) we find not only the 60° points marked by stones but also five of the intermediate 30° points as well as changes of radius, etc.

A good example of a type A circle is found at Dinnever Hill in Cornwall. It will be seen in Fig. 2 that the stones are mostly small which for our purpose has the obvious advantage of defining the outline more exactly than would have been the case with very large stones. Further examples of type A are shown in Figs. 3 and 6 Fig. 4 shows an example of Type B. It should be explained that in each case the surveys shown were plotted to a large scale. The geometrical construction was then plotted on tracing paper and

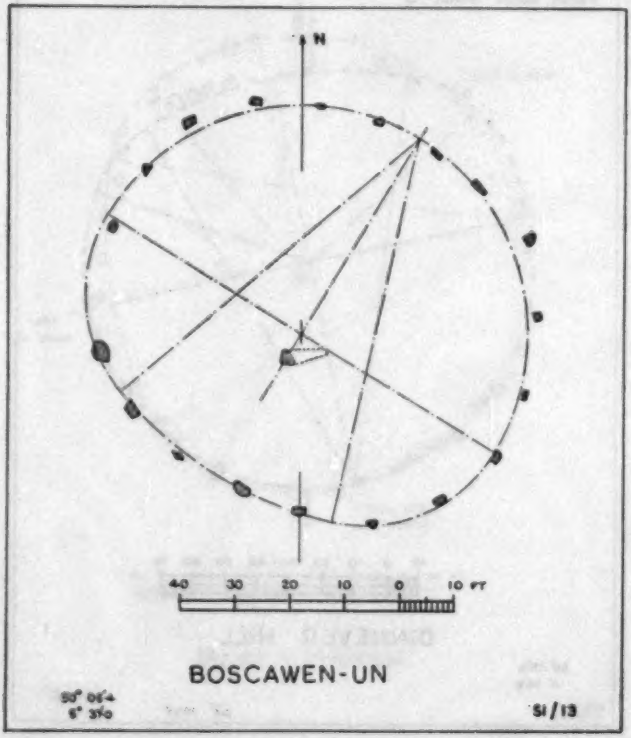

FIG. 4

moved about until the best position was found. The survey was then reduced by a precision pantograph for reproduction. Stones which are still definitely upright are shown hatched or shaded. Stones which are doubtful or fallen are shown in outline.

It is interesting to note that at Black Marsh in Shropshire (Fig. 3) the stone at one end of each axis is specially marked by having circular holes about 1½ inches diameter and 3 to 4 inches deep. This circle also illustrates a universal characteristic of circles with a centre stone. The stone is never exactly on the geometric centre. Presumably during erection and perhaps for long after a pole stood here to which the rope used for setting out was attached. The stone was usually placed close to or touching the pole.

A very interesting circle which belongs to neither type is found at Rough Tor in Cornwall (Fig. 5). It is similar to type A but the

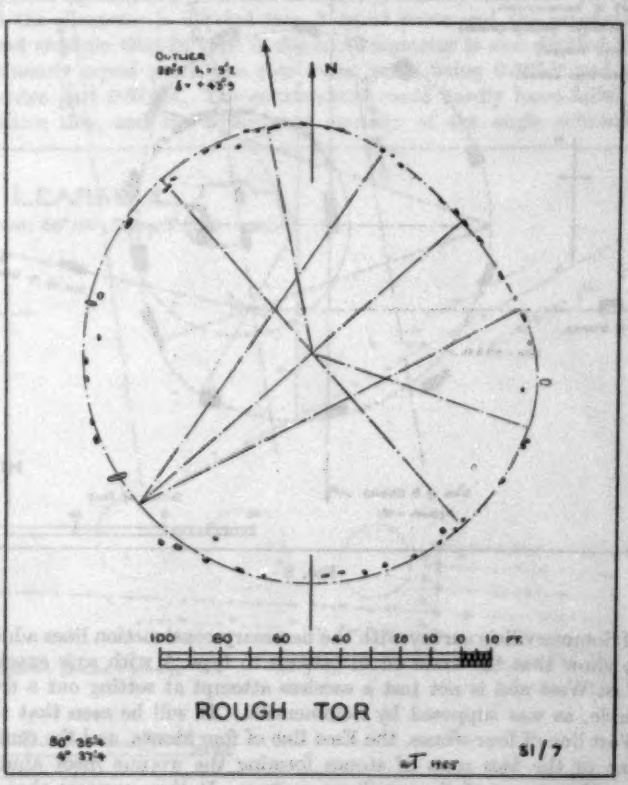

FIG. 5

radius of the small arcs is one-third of the principal diameter (as in type B) instead of one-quarter. Here again it is fortunate that the stones are small, allowing us to be quite certain as to the geometric shape. It is possible that the circle at Seascale in Cumberland belongs to this type, but no other good example is known to me.

Perhaps the most important site in Britain, and certainly the most interesting, is that at Callanish in the island of Lewis. A good survey has been twice published by Sommerville, Ref. 2 and 3, but a redetermination of the orientation shows that his azimuths should be increased by 0° 35′. Fig. 6 is taken from the inner part

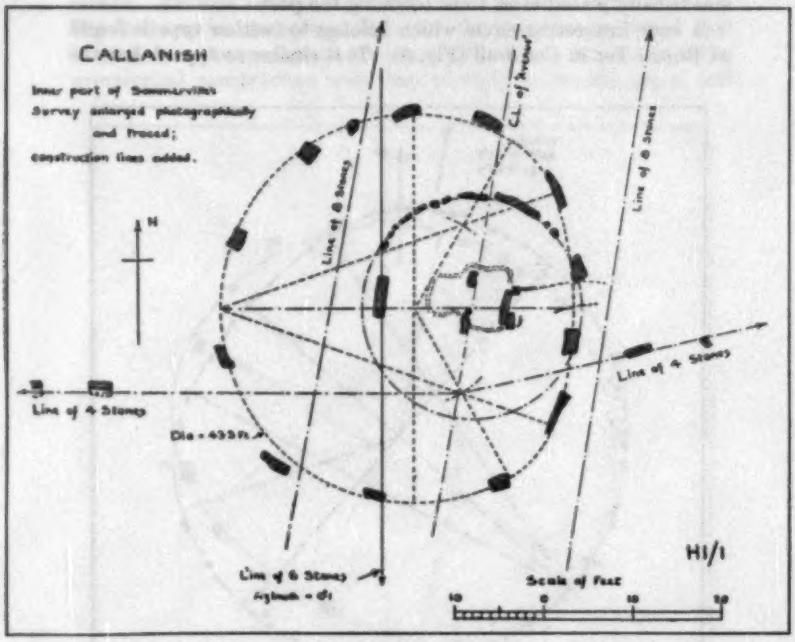

FIG. 6

of Sommerville's survey with the necessary construction lines added to show that the main circle belongs to type A with axis exactly East/West and is not just a careless attempt at setting out a true circle, as was supposed by Sommerville. It will be seen that the West line of four stones, the East line of four stones, and the centre line of the two rows of stones forming the avenue meet almost exactly at one of the auxiliary centres. It thus appears that in

some cases the constructors attached a special significance to the auxiliary centre, but in others the main centre appears to have been the important point. To give one example, at Long Meg and Her Daughters in Cumberland (type B, Ref. 1) the 12 ft. high outlier (Long Meg) *as viewed from the main construction centre* gives a declination on the horizon of 24°·2 S which was the declination of the lower limb of the mid-winter sun circ. 2000 B.C. (Ref. 4).

The megalithic fathom appears in the circle diameters, in the distance to outliers, and in the distance between circles where two are close together. Evidently it cannot also appear in the circumference and it is interesting to speculate if this was the reason for flattening the circles. Were the builders attempting to make a circle with π exactly 3? The ratio of periphery to diameter is 3·059 for type A and 2·957 for type B. The number 3 may have had a special significance; it is still considered a lucky number. In type B the diameter is divided into 3 equal parts and the interesting fact emerges that in type A the cross diameter is also divided into 3 nearly equal parts, the two outer parts being $0{\cdot}327D$ and the centre part $0{\cdot}345D$. The constructors could hardly have failed to notice this, and the consequent equality of the angle subtended

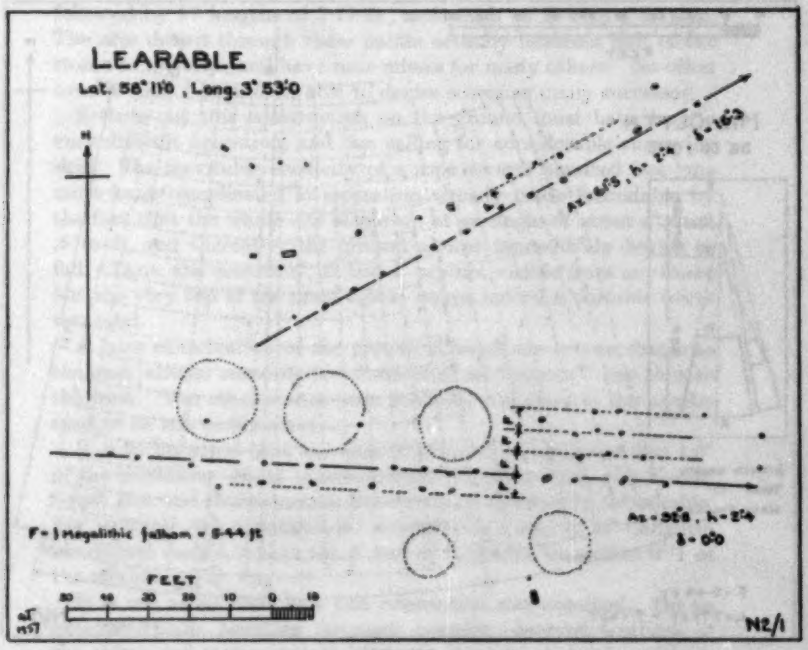

FIG. 7

by the long radius arc in the two types A and B. Thus in type A angle GDG is 38°·2 and in type B the angle LJL is 36°·9. It is not intended in this paper to describe the egg-shaped circles found in one or two localities. These again give evidence of careful setting out with circular arcs, and again the fathom appears.

Instead, it is now proposed to look at a totally different type of construction found in the Caithness and Sutherland alignments. On the high plateau on Learable Hill above Suisgill in Sutherland there are several lines of stones, some of which appear to be parallel. An accurate survey immediately reveals their purpose. In Fig. 7 we see that they fall into two groups, lines running at an azimuth of 92°·8 and lines running at 61°·5. The far off mountain horizon has at these azimuths heights of 2°·4 in both cases and this, with the azimuths, gives declinations of 0°·0 and +16°·3. The alignments thus indicate the rising points of the sun at the equinox and on a day half-way between the equinox and the solstice, sometimes called May Day. Other examples are given in Ref. 4 together with an assessment of the values of the declination (0°·6 and 16°·9) to be expected by an examination of the earth's orbit at 2000 B.C.

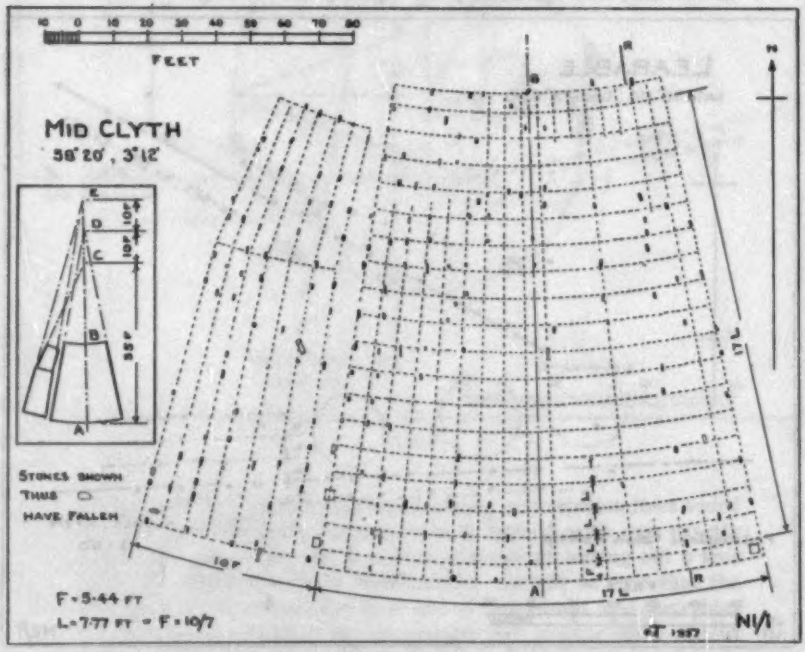

FIG. 8

At Mid Clyth there is another set of alignments of a totally different character. An accurate survey was made to a large scale and carefully studied. A reduced scale reproduction is shown in Fig. 8. This shows that the layout consists of radiating rows of stones. On a large scale plan a mean line was drawn through each row. Co-ordinates of points on these lines were then *calculated* in the neighbourhood of the intersection points so that the intersections could be plotted. It was thus found that there were three main centres, each lying on the axis of symmetry. The majority of the lines in the larger symmetrical part of the figure radiate from *D* (see inset diagram) but one marked *RR* seems to belong to the point *E*, which is the centre for the southerly part of the " annex " to the west. After making many assumptions regarding the arcs running across the figure, it was found that a possible construction of the main figure was as follows. Draw an arc through *A* with radius 65 megalithic fathoms, i.e. $65 \times 5{\cdot}44$ ft. Divide this into 17 parts ($8\frac{1}{2}$ on each side of *A*) each 7·77 ft. These points fix the radiating lines. The arcs themselves are at the same spacing with the exception of the top and bottom spaces which are each 1 fathom. Thus, starting from *A*, along the axis of symmetry set out 1 fathom followed by 17 lengths of 7·77 ft., and finish at *B* with 1 fathom. The arcs drawn through these points actually intersect half of the stones in the area, and have near misses for many others. No other construction I have been able to devise scores so many successes.

Setting out this construction on the ground must have been a very difficult operation, and one calling for considerable surveying skill. The inevitable elasticity of a rope several hundred feet long must have complicated an operation already made formidable by the fact that the whole site slopes up at an angle of about 4° from *A* to *B*, and thereafter the ground almost immediately begins to fall. Thus, the centres *C*, *D* and *E* are not visible from anywhere but the very top of the main figure, unless indeed a portable tower was used.

A later examination of the ground showed one or two stones to the east, almost certainly the remains of an " annex " like that to the west. The other stones were probably too close to the nearby road to be left undisturbed.

It is to be noted that the axis of symmetry *AB* lies within 1·5° of the meridian. Many other sites contain indications of a North/South line and in some cases the accuracy attained is remarkable. For example the correction to Sommerville's survey of Callanish mentioned earlier, makes the *S* line of megaliths lie within 0°·1 of the meridian (Fig. 6).

It is not at all clear how this orientation was obtained. On an extensive plain, bisecting the angle between observed positions of the rising and setting points of a star provides a simple method,

but in mountainous country considerable errors would arise. A shadow technique presupposes a horizontal surface and so can be ignored. At the time of the construction of these sites the pole star was nowhere near the pole, but circumpolar stars were available. To use a circumpolar star we can imagine a long plumb line from the top of a pole forming the foresight. At each elongation of the star a stake could be used to mark the backsight. We thus obtain two radiating lines at equal angles on either side of the meridian. It is possible that the figure so obtained *suggested* the construction at Mid Clyth.

It may also be noticed that the radiating lines can be used as a sundial near to apparent noon; the intervals between the lines vary from 3 minutes at midsummer to 5½ minutes at midwinter.

Enough has been said to show the importance of obtaining really accurate large scale surveys of all the sites in Britain before the tractor and the bulldozer have reduced their number. I have come across several examples which have been recently damaged and the demand to make more use of marginal land will inevitably accelerate the disappearance of the smaller and unscheduled sites. Sometimes a site which appears to be merely a few random boulders shows up quite differently on being surveyed.

In many cases the stones are buried under several feet of peat; in others only the buried stumps remain. A potent destructive force is the growth of trees on the site. In Aberdeenshire I have seen a large standing stone which had been lifted bodily out of the ground by a tree uprooted in the 1953 gales and I have seen a standing stone almost enclosed in the trunk of a tree. Is it any wonder that slight distortions occur in outlines which have survived over three thousand years? Earth movements also take place due to creep and frost. Such movement is apparent in the slight bend which can be seen in several lines in Fig. 8.

If we are to find out more about these constructions no site can be dismissed as unimportant and all should be surveyed. Although I have made plans of some four hundred sites, this is only a beginning. Digging must also be undertaken where some of the stones are buried. High class accurate surveys must be made and published on a reasonably large scale, and these should be brought together so that they are available for detailed statistical examination.

Oxford University, A.T.

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3. Somerville, B. (1912) " Prehistoric monuments in the Outer Hebrides, and their astronomical significance ", *J. Roy. Anthropological Institute*, Vol. XLII, p. 23.

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GLEANINGS FAR AND NEAR

1958. Next he should acquaint the youth with mathematicks; and to invite to it, he sould begin quickly to show him some of the more pleasant mechanical performances in mathematicks. That which is necessarily to be known to one that would study these sciences is Euclid's Elements, at least his first six books, arithmetick and trigonometry; and without one's understanding these, one may be a mechanist, but a mathematician shall he never be. For stereometry, algebra and conic sections, they require more subtlety and patience, than is to be expected from youth, neither are they of such use. . . . If the youth have a delight in problemes and theoremes, and be of ane active fancy, it will be good to hook him as much as can be to them; for this is by wise men judged a good advice for preserving a state quiet, to engage the young nobility who have active spirits, to mathematical sciences, which carrying their thoughts after them, will preserve them from ambition, and meddling with the state. But in this moderation is to be observed, lest their brains be too much stretched with these curiosities.—Gilbert Burnet, *Thoughts on Education* (written 1668, published 1761). [Per Mr. J. W. Ashley Smith.]

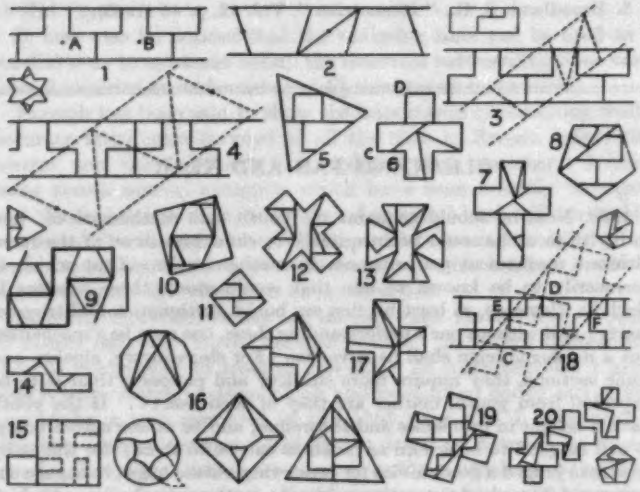
1959. On the opposite wall he could see the six-inch map of the Chaddesbourne estate, its irregular equilateral triangle outlined with a wash of red water-colour. That shape, which he had always considered as permanent and immutable as the triangular outline of England itself, no longer represented the geographical truth. The equilateral triangle had become an isosceles.—F. B. Young, *This Little World*. [Per Mr. B. J. Barnes.]

1960. They were taken to hospital and allowed home after being %-rayed and treated for cuts and bruises.—*Daily Telegraph and Morning Post*, Thursday, June 26, 1958. [Per Mr. T. Knape Smith.]

GOING ONE BETTER IN GEOMETRIC DISSECTIONS

By H. Lindgren

Methods have been described that facilitate the discovery of economical dissections [1]. They are summarized in §§ 1 and 2 that follow.



1. Each of the figures to be dissected is made an element of a strip with parallel sides; *1* and *2* show strips formed from a Latin or tau cross and a pentagon. The first is a *P-strip* (prototype parallelogram), in which all elements are the same way up. The second is a *T-strip* (prototype trapezium), in which alternate ones are inverted. Points such as *A* and *B* in *1* are called *congruent*, a term copied from Whittaker and Watson's *Modern Analysis*, p. 430.

Having drawn on tracing paper two such strips of elements with the same area (3 sq. in., say), superpose them, as in *3*, at an angle such that the edges of one strip pass through congruent points in the other, if this is a *P*-strip, or through the midpoints of the other (shown by dots in *2* and *3*), if it is a *T*-strip. The lines in the area common to the strips give a dissection of the one figure into the other, and by translating one strip relatively to the other you can find a superposition in which the common area has fewest pieces. The dissection derived from *3* is shown in *7* and *8*.

Dissections obtained thus can be classified according to the strips

used. They include *PP*, *PT* (e.g. *3*), *TT*1 in which the common area is equal to that of each figure, and *TT*2 in which it is twice as much. The last kind is illustrated in *4*, *5*, *6*, which show how the lines in half the common area give the dissection sought.

Freedom in translating the strips is greatest in *PP*-dissections; one has only to keep the angle between strips constant. In *PT*-dissections the midpoints of the *T*-strip must stay on the edges of the other. This applies to both strips in *TT*-dissections, precluding continuous translation; but there is a choice of midpoints, and the oblique strip can slope down as well as up.

The procedure can be applied to solid dissections, if three-dimensional tracing paper is available. The first step is to get strips of the same height.

2. Dissections can also be found from tessellations. If the figures to be dissected can be made elements of tessellations having the same pattern of congruent points, one merely superposes the tessellations so that points congruent in the one are congruent in the other. In this way you can find such dissections as an 18-piece one of three hexagons into seven.

A number of figures can be made elements of tessellations with a square pattern of congruent points. Thus a Maltese cross, rearranged as in *9*, is an element of a square tessellation, and from the latter one finds the dissection into a square shown in *10*. Similarly it is mere routine to find the dissection of Greek and Maltese crosses shown in *12* and *13*.

Freedom in translating superposed tessellations corresponds more or less to the case of strips, being greatest for instance when both have the same period parallelogram with all elements the same way up. The restrictions necessary in any particular case show themselves readily enough in actual trial.

A further use of tessellations is in seeking strips suitable for superposition as in § 1. They are obtained by drawing on the tessellation a pair of congruent lines (i.e. parallel lines through congruent points). This is exemplified in § 3.

3. The only published dissection of a Greek cross and an equilateral triangle that I know of requires six pieces [2]. In superposing strips as in *4* we have gone one better than our predecessors. Again, the only published dissection of a Maltese cross and a square that I know of requires 13 pieces [3*a*]. In superposing tessellations to get *10* we have gone five better. It will also be found, on superposing a strip of squares on *1*, that a Latin cross can be dissected into a square in five pieces, not six, and that there is a way of opening out a cube, shown in *14* and *11*, such that the plane figure obtained can be dissected into a square in four pieces, not five [3*b*].

One Greek cross has been dissected into three in 13 pieces [4]. Going one better in this case was not so easy.

In *18* the full-line three-cross strip is to have superposed on it a one-cross strip. This will be found from a Greek-cross tessellation by drawing on it two lines parallel to *CD* in *6*; but their positions are left undetermined for the moment, and we superpose the whole tessellation on the three-cross strip. Trial shows that it is best to have *C* and *D* on the edges of the strip where shown in *18*. Likewise it is found best to have the edges of the one-cross strip pass through the congruent points *E*, *F*. This determines the positions of the edges of the one-cross strip, and the *PP*-dissection obtained has 12 pieces. It is shown in *19* and *20*.

4. Quite a lot of published dissections could have been found without difficulty by the methods explained in §§ 1 and 2. Thus Dudeney's triangle-square dissection [5, 6*a*] is a *TT*2 similar to *4*; Travers's octagon-square dissection [7*a*] is obtained by superposing the 4.8^2 tessellation [6*b*, 7*b*] on the square tessellation with element as in *15* (an example of "completing the tessellation"); and my six-piece dissections of a dodecagon into a square and into a Greek cross [1, 8, *16*] were found by superposing tessellations. Sometimes better dissections can be found by applying these methods. But usually we do not go one better than our predecessors, for so many dissections can be attempted that they have not ventured on. Try instead to improve on the numbers of pieces given in the table below, or to fill in the gaps, or go on to three-figure dissections such as *17*.

4	4									
5	6	6								
6	5	5	7							
7	10	9	11	11						
8	8	5	9	9	13					
12	8	6								
G	5	4	7	7	12	9	6			
L	5	5	8	6		8		7		
M		8						9		
S		6						8		9
	3	4	5	6	7	8	12	*G*	*L*	*M*

(A number *n* on the left and below denotes regular *n*-gon; *G*, *L*, *M* denote Greek, Latin, and Maltese crosses; and *S* denotes the symbol of evil memory depicted in *15*.)

Patent Office, Canberra H.L.

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MAESTRO PUZZLES

By J. E. Reeve and J. A. Tyrrell

A few years ago there appeared on the commercial market a certain puzzle called "Maestro". This consists of packing into a 6×10 rectangle the twelve shapes depicted in Fig. 1, (where they

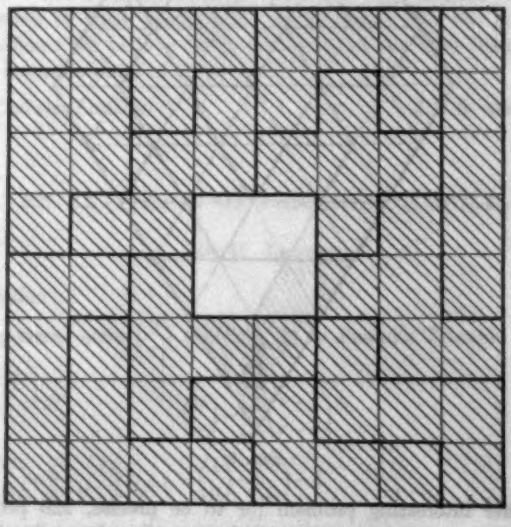

Fig. 1

are shown packed into an annular region bounded by two concentric squares). It is also possible to pack these shapes into other

simple geometrical figures. The aesthetic attraction of this problem lies in the fact that the twelve shapes concerned form a complete set in the sense that any other shape consisting of five abutting squares is congruent to one of them.

The purpose of this note is to remark that the twelve non-congruent shapes that can be formed by using, in place of five equal squares, six equal equilateral triangles may be packed into a 60 degree rhombus. Ignoring additional solutions obtained by reflections in either diagonal of the rhombus, over forty solutions are at present known to us, of which one is shown in Fig. 2.

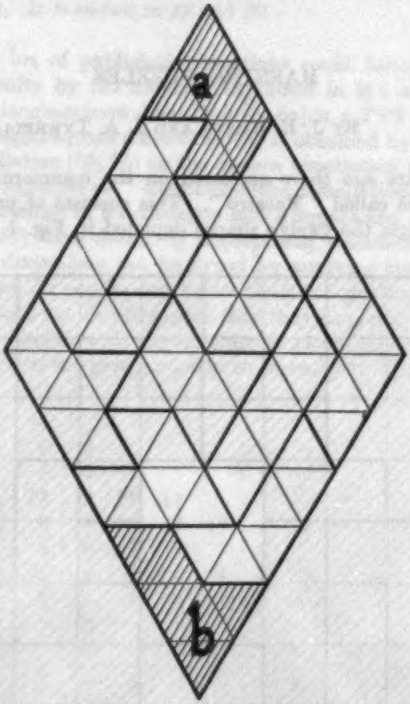

FIG. 2

Another interesting problem (or to be precise, 220 problems!) consists in arranging any nine of these twelve shapes in the form of a regular hexagon. One such arrangement is shown in Fig. 3. Of these 220 problems 90 are readily shown to admit no solution; these correspond precisely to the sets of nine shapes which include

one (but not both) of the shapes marked " a " and " b " in Fig. 2. To see this it suffices to remark that if the hexagon is coloured in chess board fashion as in Fig. 3 then all the shapes except " a "

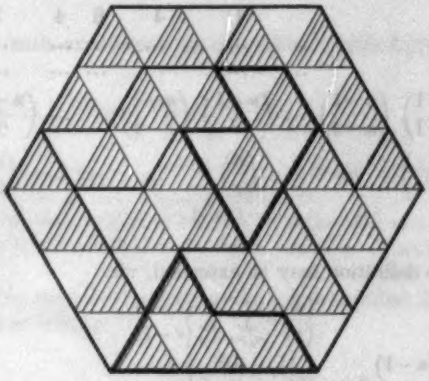

FIG. 3

and " b " cover equal numbers of white and shaded triangles, irrespective of their position in the hexagon. Of the remaining 130 problems we believe 22 to be insoluble but have been unable to find a simple proof of this.

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AN EXTENSION OF PASCAL'S TRIANGLE

BY H. LEVITT

Introduction

The general term $\binom{n}{r}$ (see Fig. 1) in Pascal's Triangle may be defined as:

$$\binom{n}{r} = \sum_{i=r}^{n} \binom{i-1}{r-1} \tag{1a}$$

where $\binom{n}{0} = \binom{n-1}{0} = \ldots = \binom{1}{0} = \binom{0}{0} = 1$

n and r are positive integers, where $n \geqslant r$.

$\binom{n}{r}$ is equivalent to the binomial coefficient, and is equal to $n!/[r!\,(n-r)!]$ (1b)

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 1 & & \\
 & & & & & & & 1 & 1 & & \\
 & & & & & & 1 & 2 & 1 & & \\
 & & & & & 1 & 3 & 3 & 1 & & \\
 & & & & 1 & 4 & 6 & 4 & 1 & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & \binom{n-1}{n-1} & \binom{n-1}{n-2} & \cdots & \binom{n-1}{r} & \binom{n-1}{r-1} & \cdots & \cdots & \binom{n-1}{0} & \\
\binom{n}{n} & & \binom{n}{n-1} & \cdots & \cdots & \binom{n}{r} & \cdots & \cdots & \cdots & & \binom{n}{0}
\end{array}
$$

FIG. 1

The above definition may be extended, viz.:

$$\binom{n}{r} = \sum_{n_1=r}^{n} \sum_{i=r}^{n_1} \binom{i-1}{r-1} \tag{2a}$$

where $\binom{n}{0} = \binom{n-1}{0} = \ldots = \binom{1}{0} = \binom{0}{0} = 1$

n and r are positive integers, where $n \geqslant r$.

In general, the term ${}^{n}d_r$ is defined as:

$${}^{n}d_r = \sum_{n_{M-1}=r}^{n} \sum_{n_{M-2}=r}^{n_{M-1}} \cdots \sum_{n_1=r}^{n_2} \sum_{i=r}^{n_1} {}^{i-1}d_{r-1} \tag{2b}$$

there being M summations.

Several relations regarding terms of the above types are derived.

2. *Theorem A.*

If we have any number ${}^{n}b_r$ such that

$${}^{n}b_r = \sum_{i=r}^{n} {}^{i-1}b_{r-1}$$

n and r are positive integers, where $n \geqslant r$, then

$${}^{n}b_r = \left[\binom{m-1}{m-1} \cdot {}^{n-m}b_{r-m} + \binom{m}{m-1} \cdot {}^{n-m-1}b_{r-m} + \cdots + \binom{m+n-r-1}{m-1} \cdot {}^{r-m}b_{r-m} \right]$$

where m is any integer such that $1 \leqslant m \leqslant r$.

Proof.

The statement

$${}^{n}b_r = [{}^{n-1}b_{r-1} + {}^{n-2}b_{r-1} + \ldots + {}^{r-1}b_{r-1}] \tag{3}$$

may be regarded as an "expansion" of ${}^{n}b_r$. If each term on the

R.H.S. of equation 3 is similarly expanded, then the following expression will be obtained:

$$ {}^{n}b_{r} = [{}^{n-2}b_{r-2} + 2 \cdot {}^{n-3}b_{r-2} + 3 \cdot {}^{n-4}b_{r-2} + \ldots + (n-r+1) \cdot {}^{r-2}b_{r-2}] \tag{4} $$

After m such expansions an expression of the following form is obtained:

$$ {}^{n}b_{r} = [{}^{m-1}h_{m-1} \cdot {}^{n-m}b_{r-m} + {}^{m}h_{m-1} \cdot {}^{n-m-1}b_{r-m} + \ldots + {}^{m+n-r-1}h_{m-1} \cdot {}^{r-m}b_{r-m}] \tag{5} $$

where ${}^{m-1}h_{m-1}$, ${}^{m}h_{m-1}$, ..., ${}^{m+n-r-1}h_{m-1}$ indicate the coefficients of ${}^{n-m}b_{r-m}$, ${}^{n-m-1}b_{r-m}$, ..., ${}^{r-m}b_{r-m}$ respectively.

The convention used in this notation is that the sum of upper and the sum of lower indices in any term equals $n-1$ and $r-1$ respectively.

In carrying out the above expansions, it is seen that the coefficients are related as follows:

$$ {}^{m+q}h_{m} = \sum_{i=m}^{m+q} {}^{i-1}h_{m-1} $$

and from inspection (equation (3)),

$$ {}^{0}h_{0} = {}^{1}h_{0} = \ldots = {}^{n-r}h_{0} = 1 $$

hence

$$ {}^{m+q}h_{m} = \binom{m+q}{m} $$

Equation (5) thus becomes

$$ {}^{n}b_{r} = \left[\binom{m-1}{m-1} \cdot {}^{n-m}b_{r-m} + \binom{m}{m-1} \cdot {}^{n-m-1}b_{r-m} + \ldots + \binom{m+n-r-1}{m-1} \cdot {}^{r-m}b_{r-m} \right] \tag{6} $$

In order to derive the above expression, at least one expansion must be known; also, we cannot expand ${}^{n}b_{r}$ more than r times, hence m is any integer such that $1 \leqslant m \leqslant r$.

Corollary.

$$ \binom{n}{r} = \binom{m-1}{m-1}\binom{n-m}{r-m} + \binom{m}{m-1}\binom{n-m-1}{r-m} + \ldots + \binom{m+n-r-1}{m-1}\binom{r-m}{r-m} \quad (1 \leqslant m \leqslant r) \tag{7} $$

This is obvious since $\displaystyle \binom{n}{r} = \sum_{i=r}^{n} \binom{i-1}{r-1}$.

It is interesting to compare this result with Vandermonde's Theorem which, using this notation, may be stated as:

$$\binom{n}{r}=\binom{m}{0}\binom{n-m}{r}+\binom{m}{1}\binom{n-m}{r-1}+\ \dots$$
$$+\binom{m}{r-1}\binom{n-m}{1}+\binom{m}{r}\binom{n-m}{0}$$

The above theorem may also be proved in a way similar to Vandermonde's method.

3. *Theorem B.*

Given a number ${}^{n}b_{r}$ such that

$${}^{n}b_{r}=\left[\binom{M-1}{M-1}\cdot{}^{n-1}b_{r-1}+\binom{M}{M-1}\cdot{}^{n-2}b_{r-1}+\ \dots\right.$$
$$\left.+\binom{M+n-r-1}{M-1}\cdot{}^{r-1}b_{r-1}\right]$$

where n and r are $+$ve integers, and $n\geqslant r$. M is an integer >0, then

$${}^{n}b_{r}=\left[\binom{mM-1}{mM-1}\cdot{}^{n-m}b_{r-m}+\binom{mM}{mM-1}\cdot{}^{n-m-1}b_{r-m}+\ \dots\right.$$
$$\left.+\binom{mM+n-r-1}{mM-1}\cdot{}^{r-m}b_{r-m}\right]$$

where m is any integer such that $1\leqslant m\leqslant r$.

Also, if ${}^{n-r}b_{0}={}^{n-r-1}b_{0}=\ \dots\ ={}^{1}b_{0}={}^{0}b_{0}=1$, then

$${}^{n}b_{r}=\binom{n+(M-1)r}{Mr}$$

Proof.

$${}^{n}b_{r}=\left[\binom{M-1}{M-1}\cdot{}^{n-1}b_{r-1}+\binom{M}{M-1}\cdot{}^{n-2}b_{r-1}+\ \dots\right.$$
$$\left.+\binom{M+p-1}{M-1}\cdot{}^{n-p-1}b_{r-1}+\ \dots\ +\binom{M+n-r-1}{M-1}\cdot{}^{r-1}b_{r-1}\right]$$

Expanding each of the above terms in turn yields an expression of the form

$${}^{n}b_{r}=[{}^{1}h_{1}\cdot{}^{n-2}b_{r-2}+{}^{2}h_{1}\cdot{}^{n-3}b_{r-2}+\ \dots\ +{}^{p+1}h_{1}\cdot{}^{n-p-2}b_{r-2}+\ \dots$$
$$+{}^{n-r+1}h_{1}\cdot{}^{r-2}b_{r-2}]$$

where the coefficient of the general term ${}^{n-p-2}b_{r-2}$ is:

$${}^{p+1}h_{1}=\binom{M-1}{M-1}\binom{M+p-1}{M-1}+\binom{M}{M-1}\binom{M+p-2}{M-1}+\ \dots$$
$$+\binom{M+p-1}{M-1}\binom{M-1}{M-1}$$

whence by equation (7) this becomes

$$ {}^{p+1}h_1 = \binom{2M+p-1}{2M-1} $$

After the third expansion

$$ {}^{n}b_r = [{}^{2}h_2 \,.\, {}^{n-2}b_{r-2} + {}^{3}h_2 \,.\, {}^{n-4}b_{r-2} + \ldots + {}^{p+2}h_2 \,.\, {}^{n-p-2}b_{r-2} + \ldots + {}^{n-r+2}h_2 \,.\, {}^{r-2}b_{r-2}] $$

where the coefficient of the general term ${}^{n-p-2}b_{r-2}$ is:

$$ {}^{p+2}h_2 = \binom{2M-1}{2M-1}\binom{M+p-1}{M-1} + \binom{2M}{2M-1}\binom{M+p-2}{M-1} + \ldots + \binom{2M+p-1}{2M-1}\binom{M-1}{M-1} = \binom{3M+p-1}{3M-1} $$

whence, after m expansions,

$$ {}^{n}b_r = [{}^{m-1}h_{m-1} \,.\, {}^{n-m}b_{r-m} + \ldots + {}^{p+m-1}h_{m-1} \,.\, {}^{n-p-m}b_{r-m} + \ldots + {}^{n-r+m-1}h_{m-1} \,.\, {}^{r-m}b_{r-m}] $$

where the coefficient of the general term ${}^{n-p-m}b_{r-m}$ is, by induction,

$$ {}^{p+m-1}h_{m-1} = \binom{mM+p-1}{mM-1} $$

i.e.

$$ {}^{n}b_r = \left[\binom{mM-1}{mM-1} \,.\, {}^{n-m}b_{r-m} + \binom{mM}{mM-1} \,.\, {}^{n-m-1}b_{r-m} + \ldots + \binom{mM+p-1}{mM-1} \,.\, {}^{n-m-p}b_{r-m} + \ldots + \binom{mM+n-r-1}{mM-1} \,.\, {}^{r-m}b_{r-m}\right] \tag{8} $$

where m is any integer such that $1 \leqslant m \leqslant r$.

If we expand r times,

$$ {}^{n}b_r = \left[\binom{Mr-1}{Mr-1} \,.\, {}^{n-r}b_0 + \binom{Mr}{Mr-1} \,.\, {}^{n-r-1}b_0 + \ldots + \binom{n+(M-1)r-1}{Mr-1} \,.\, {}^{0}b_0\right] $$

Now ${}^{n-r}b_0 = {}^{n-r-1}b_0 = \ldots = {}^{1}b_0 = {}^{0}b_0 = 1$ by hypothesis.

$$ \therefore \quad {}^{n}b_r = \left[\binom{Mr-1}{Mr-1} + \binom{Mr}{Mr-1} + \ldots + \binom{n+(M-1)r-1}{Mr-1}\right] $$

i.e.

$$ {}^{n}b_r = \binom{n+(M-1)r}{Mr} \tag{9} $$

Corollary.

$$ \binom{n}{r} = \binom{2m-1}{2m-1}\binom{n-m}{r-m} + \binom{2m}{2m-1}\binom{n-m-1}{r-m} + \ldots + \binom{2m+n-r-1}{2m-1}\binom{r-m}{r-m} $$

Proof.

We have $\binom{n}{r} = \sum_{i=r}^{n} \binom{i-1}{r-1} + \sum_{i=r}^{n-1} \binom{i-1}{r-1} + \ldots + \sum_{i=r}^{r} \binom{i-1}{r-1}$

$$= \binom{n-1}{r-1} + 2 \cdot \binom{n-2}{r-1} + \ldots + (n-r+1) \cdot \binom{r-1}{r-1}$$

$$= \binom{1}{1}\binom{n-1}{r-1} + \binom{2}{1} \cdot \binom{n-2}{r-1} + \ldots + \binom{n-r+1}{1} \cdot \binom{r-1}{r-1}.$$

Hence, by the above theorem, where $M = 2$,

$$\binom{n}{r} = \binom{2m-1}{2m-1}\binom{n-m}{r-m} + \binom{2m}{2m-1}\binom{n-m-1}{r-m} + \ldots$$

$$+ \binom{2m+n-r-1}{2m-1} \cdot \binom{r-m}{r-m} \quad (1 \leqslant m \leqslant r) \quad (10)$$

Corollary.

$$\binom{n}{r} = \binom{n+r}{2r}$$

Proof.

For the case when $m = r$, by equation (10),

$$\binom{n}{r} = \binom{2r-1}{2r-1} \cdot \binom{n-r}{0} + \binom{2r}{2r-1} \cdot \binom{n-r-1}{0} + \ldots + \binom{n+r-1}{2r-1} \cdot \binom{0}{0}$$

But $\binom{n-r}{0} = \binom{n-r-1}{0} = \ldots = \binom{1}{0} = \binom{0}{0} = 1$, by definition,

$$\therefore \quad \binom{n}{r} = \binom{2r-1}{2r-1} + \binom{2r}{2r-1} + \ldots + \binom{n+r-1}{2r-1}$$

i.e.
$$\binom{n}{r} = \binom{n+r}{2r} \quad (11)$$

4. *The General Case.*

We define a number ${}^{n}d_{r}$ such that

$${}^{n}d_{r} = \sum_{{}^{n}M-1={}^{n}M-2=r}^{n} \sum^{{}^{n}M-1} \ldots \sum_{n_1=r}^{n_2} \sum_{i=r}^{n_1} {}^{i-1}d_{r-1} \quad (M \text{ summations}) \quad (12)$$

or, using the following notation for convenience, this definition may be written as:

$$\left[\begin{matrix} n \\ r \end{matrix}\right]_0^M = \left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right]_M^M$$

where n and r have the same meaning as before; the lower index represents the number of summations; the upper index represents the " degree " of the number. e.g.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{r-1}{r-1} = \sum_{i=r}^{n} \binom{i-1}{r-1}$$

may be written

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^1 = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_0^1 + \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_0^1 + \dots + \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_0^1 = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_1^1$$

or,

$$\binom{n}{r} = \sum_{i=r}^{n} \binom{i-1}{r-1} + \sum_{i=r}^{n-1} \binom{i-1}{r-1} + \dots + \sum_{i=r}^{r} \binom{i-1}{r-1} = \sum_{n_1=r}^{n} \sum_{i=r}^{n_1} \binom{i-1}{r-1}$$

may be written

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^2 = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_1^2 + \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_1^2 + \dots + \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_1^2 = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_2^2$$

5. *Theorem C.*

If we define a number $\begin{bmatrix} n \\ r \end{bmatrix}_0^M$ such that

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^M = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M$$

then

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^M = \binom{m-1}{m-1} \cdot \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{M-m}^M + \binom{m}{m-1} \cdot \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{M-m}^M + \dots + \binom{m+n-r-1}{m-1} \cdot \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_{M-m}^M$$

also

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^M = \binom{mM-1}{mM-1} \cdot \begin{bmatrix} n-m \\ r-m \end{bmatrix}_0^M + \binom{mM}{mM-1} \cdot \begin{bmatrix} n-m-1 \\ r-m \end{bmatrix}_0^M + \dots + \binom{mM+n-r-1}{mM-1} \cdot \begin{bmatrix} r-m \\ r-m \end{bmatrix}_0^M \quad (1 \leqslant m \leqslant r)$$

and if $\begin{bmatrix} n-r \\ 0 \end{bmatrix}_0^M = \begin{bmatrix} n-r-1 \\ 0 \end{bmatrix}_0^M = \dots = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0^M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_0^M = 1$, then

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^M = \begin{bmatrix} n+(M-1)r \\ Mr \end{bmatrix}_0^1 \quad \text{i.e.} \quad \binom{n+(M-1)}{Mr}.$$

Proof.

By definition

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{M-1}^M + \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{M-1}^M + \dots + \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_{M-1}^M$$

expanding again,

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = 1 \cdot \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{M-2}^M + 2 \cdot \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{M-2}^M + 3 \cdot \begin{bmatrix} n-3 \\ r-1 \end{bmatrix}_{M-2}^M + \dots + (n-r+1) \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_{M-2}^M$$

After m such expansions, following the method used to prove theorem A:

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{m-1}{m-1} \cdot \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{M-m}^M + \binom{m}{m-1} \cdot \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{M-m}^M + \dots + \binom{M+n-r-1}{m-1} \cdot \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_{M-m}^M \quad (1 \leqslant m \leqslant r) \quad (13)$$

After M such expansions,

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{M-1}{M-1}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_0^M + \binom{M}{M-1}\begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_0^M + \dots + \binom{M+n-r-1}{M-1}\begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_0^M$$

whence, by theorem B:

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{mM-1}{mM-1} \cdot \begin{bmatrix} n-m \\ r-m \end{bmatrix}_0^M + \binom{mM}{mM-1} \cdot \begin{bmatrix} n-m-1 \\ r-m \end{bmatrix}_0^M + \dots + \binom{mM+n-r-1}{mM-1} \cdot \begin{bmatrix} r-m \\ r-m \end{bmatrix}_0^M \quad (1 \leqslant m \leqslant r) \quad (14)$$

For the case when $m = r$:

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{Mr-1}{Mr-1} \cdot \begin{bmatrix} n-r \\ 0 \end{bmatrix}_0^M + \binom{Mr}{Mr-1} \cdot \begin{bmatrix} n-r-1 \\ 0 \end{bmatrix}_0^M + \dots + \binom{Mr+n-r-1}{Mr-1} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}_0^M$$

$$= \binom{Mr-1}{Mr-1} + \binom{Mr}{Mr-1} + \dots + \binom{Mr+n-r-1}{Mr-1}$$

since $\begin{bmatrix} n-r \\ 0 \end{bmatrix}_0^M = \begin{bmatrix} n-r-1 \\ 0 \end{bmatrix}_0^M = \dots = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_0^M = 1$ by hypothesis.

$$\therefore \quad \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{n+(M-1)r}{Mr}$$

i.e.

$$\begin{bmatrix} n \\ r \end{bmatrix}_0^M = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_M^M = \binom{n+(M-1)r}{Mr} = \begin{bmatrix} n+(M-1)r \\ Mr \end{bmatrix}_0^1 \quad (15)$$

6. *Application.*

It may be shown that the transfer functions of particular types of repetitive circuits may be expressed in terms of these numbers.

For example, a simple ladder network having an inductance L in the series branch and a capacitance C in the shunt branch, the last shunt branch being short circuited, has an input impedance of the form:

$$Z_n = j\omega L \frac{N_n}{D_n} \quad (\omega = \text{angular frequency})$$

where $D_n = \sum_{r=0}^{n} \binom{n}{r} a^r = \sum_{r=0}^{n} \binom{n+r}{2r} a^r$

and $N_n = \sum_{r=0}^{n-1} \binom{n+r}{2r+1} a^r$

where $a = -\omega^2 LC$

Since the numerator and denominator are polynomials in a having coefficients of the binomial type, they are relatively easy to factorise. E.g. it may be shown that D_n is the coefficient of x^n in the expansion of $\dfrac{1-x}{(1-x)^2 - ax}$ according to the binomial theorem, and $D_n = 0$ when

$$-a = 2(1-\cos\theta), \quad \theta = \frac{\pi}{2n+1}, \frac{3\pi}{2n+1}, \cdots, \frac{(2n+1)\pi}{2n+1}$$

It is thus seen that these numbers do have a possible practical application. Their use, however, is limited; a more general approach to the analysis of repetitive circuits is provided by the use of matrices.

7. *Acknowledgements.*

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H.L.

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ON THE RELATION BETWEEN REAL EUCLIDEAN AND COMPLEX PROJECTIVE GEOMETRY

By E. C. Zeeman

We discuss the deduction of real euclidean theorems from complex projective theorems, and the reverse process: when is it permissible, in proving a complex projective theorem, to project two given points into the circular points at infinity and use real euclidean methods? We emphasise the difference between the real and the complex, and the consequences of using real numbers to define euclidean geometry, and complex numbers to define projective geometry.

A *geometry* is the study of properties of loci in a given space, which are invariant under a given group of transformations of the space.

The *space* concerned is a set of points, and the loci are certain admissible subsets. The space can be described in two ways, either axiomatically, or by means of a number system. In the first description points and lines etc. are undefined objects, but the sets of points and lines etc. are sets obeying certain axioms. In the second description a *point* in n-dimensional space *is* an n-tuple (or an equivalence class of $(n+1)$-tuples) of numbers belonging to some given number field; a *locus* is the set of points satisfying one or more polynomial equations (or homogeneous polynomial equations) with coefficients in the field. For instance a *line* in two dimensions is the set of points satisfying a linear equation. To illustrate the connection between euclidean and projective geometry we choose the second description, and for simplicity we confine ourselves to two dimensions.

1. Real euclidean geometry

Euclidean geometry is defined over the field of real numbers. The real numbers are used because the spirit of euclidean geometry is to describe the world we live in. The *euclidean plane* E is the set of all pairs of real numbers (x, y). The *euclidean group* G_E is the set of all translations and rotations and products thereof:

$$(x, y) \to (x \cos\theta + y \sin\theta + h,\ -x \sin\theta + y \cos\theta + k),$$

where θ, h, k are real constants and $0 \leqslant \theta < 2\pi$. *Euclidean geometry* is the study of properties in E invariant under G_E.

2. Complex projective geometry

Projective geometry should always take place over an algebraically closed number field, because the spirit of projective geometry is to avoid exceptions (such as a circle meeting some lines but not others). Classical projective geometry always took place over the complex

number field, but in modern mathematics this is perhaps more the exception than the rule, for it is customary to use any algebraically closed field. However, if we are to compare projective geometry with euclidean geometry, we must use the complex field, because this is the unique closure of the real field. Let us therefore define the complex projective plane:

Consider the set of triples of complex numbers (a, b, c), excluding $(0, 0, 0)$. Define an equivalence relation on this set by

$$(a, b, c) \sim (a', b', c')$$

if there exists a complex number $t \neq 0$ such that $a' = ta$, $b' = tb$, $c' = tc$. A *point* is defined to be an equivalence class, and the point containing the triple (a, b, c) is denoted by $\{a, b, c\}$. The aggregate of such points (or equivalence classes) is called the *complex projective plane* P. A linear automorphism, or 1–1 algebraic correspondence, of P onto itself is called a *projectivity*:

$$\{a, b, c\} \to \{m_{11}a + m_{12}b + m_{13}c,\ m_{21}a + m_{22}b + m_{23}c,\ m_{31}a + m_{32}b + m_{33}c\},$$

where the m_{ij} are complex constants, and $\det(m_{ij}) \neq 0$. (Notice that the same projectivity is given if the m_{ij} are altered in proportion, and so G_P is isomorphic to the quotient group M/N, where M is the group of all non-singular complex 3×3 matrices, and N is the subgroup of all non-zero multiples of the unit matrix.) The *projective group* G_P is the set of all such projectivities. *Projective geometry* is the study of properties in P invariant under G_P.

3. Extended complex euclidean geometry

The link between E and P is achieved by means of a hybrid geometry in between, called the extended complex euclidean geometry. Let Q denote the extended complex euclidean plane, which is defined as follows. There are four types of points in this plane (as opposed to the planes E and P, in each of which there is only one type of point).

(i) Finite real points. Let E be the set of all real pairs (x, y), as before.

(ii) Finite non-real points. Let C be the set of all complex pairs (x, y), of which not both are real.

(iii) Infinite real points. Let E^∞ be the set of all equivalence classes $\{x, y\}$ of real pairs, under the equivalence relation $(x, y) \sim (tx, ty)$, where t is real non-zero, and $(0, 0)$ is excluded.

(iv) Infinite non-real points. Let C^∞ be the set of all equivalence classes $\{x, y\}$ of complex pairs, not containing any real pair, under the equivalence relation $(x, y) \sim (tx, ty)$, where t is complex non-zero.

Then $Q = E \cup C \cup E^\infty \cup C^\infty$ is the *extended complex Euclidean plane.*

Since euclidean motions are involved, the group G_Q is defined to be the same as G_E, and a particular transformation operates as follows:

$$(x, y) \to (x \cos\theta + y \sin\theta + h,\ -x \sin\theta + y \cos\theta + k),$$

for points in $E \cup C$,

$$\{x, y\} \to \{x \cos\theta + y \sin\theta,\ -x \sin\theta + y \cos\theta\},$$

for points in $E^\infty \cup C^\infty$,

where θ, h, k are real constants and $0 \leqslant \theta < 2\pi$. *Extended complex geometry* is the study of properties in Q invariant under G_Q.

4. Other related geometries

So far we have defined three different planes, the first three listed below. We mention in passing two other planes, (4) and (5), which are useful in other contexts (in particular in affine geometry and topology, respectively).

(1) The euclidean plane E.
(2) The projective plane P.
(3) The extended complex euclidean plane Q.
(4) The complex euclidean plane $E \cup C$.
(5) The extended real euclidean plane $E \cup E^\infty$.

All the planes except for (2) have the same euclidean group of transformations G_E.

It is possible to obtain many other geometries by enlarging the group of transformations. Then as a consequence there will be fewer invariant properties, but these few may stand out in sharper contrast. For instance consider the effect of adding to G_E various other transformations of E: if we add reflections, then orientation is no longer an invariant, but distance is preserved. If, further, we add magnification about any point, then distance is lost but perpendicularity is preserved. If, further, we add affine transformations, then perpendicularity is lost but linearity is preserved. If, further, we add topological transformations, then linearity is lost but connectedness is preserved. If, further, we add arbitrary transformations, then all geometrical structure is lost, and only set-theoretical properties such as cardinality are preserved. However, we do not dwell upon these possibilities, because our main objective is to discuss the connection between (1) and (2) by means of (3).

5. Special points in Q

Notice that each transformation in G_Q preserves the four types of points, E, C, E^∞, C^∞. Therefore the *type* of a point is an invariant

property, and is consequently, by definition, an essential feature of the geometry, the extended complex euclidean geometry.

There are also two points in C^∞, namely $I=\{1, i\}$ and $J=\{1, -i\}$, which are left invariant by all the transformations in G_Q. We call them the circular points at infinity. They are the only two points in Q which are left invariant by G_Q, and are therefore, by definition, special points in the geometry. This property of invariance is the basic underlying reason for their significance. Notice that, of the five planes mentioned in the last section, only Q contains the circular points at infinity.

6. The link between E and Q

By definition E is contained in Q, and $G_E=G_Q$. Therefore, given a locus L_Q in Q, we may take the intersection $L_E=L_Q\cap E$ and obtain a locus in E, which we call the *real cross-section* of L_Q. Any property of L_Q invariant under G_Q is *a fortiori* invariant under G_E, and so yields a property of L_E invariant under G_E. Hence a Q-theorem implies an E-theorem. In particular, if L_Q is a conic through I and J, then L_E is a circle; this property of I and J is of course responsible for their designation as circular points at infinity.

7. The link between P and Q

We may establish a linear isomorphism $Q\to P$ by

$$\begin{aligned}(x, y)&\to\{x, y, 1\}, \quad \text{for points in } E\cup C,\\ \{x, y\}&\to(x, y, 0\}, \quad \text{for points in } E^\infty\cup C^\infty.\end{aligned}$$

In what follows it will be convenient to let f denote the inverse isomorphism $P\to Q$, which transforms

$$\begin{aligned}\{a, b, c\}&\to(a/c, b/c), \quad \text{if } c\neq 0,\\ \{a, b, 0\}&\to\{a, b\}, \quad \text{if } c=0.\end{aligned}$$

Under this isomorphism, G_Q is contained as a subgroup in G_P in the following sense: to each transformation g_Q in G_Q there corresponds a projectivity

$$g_P=\begin{pmatrix}\cos\theta & \sin\theta & h\\ -\sin\theta & \cos\theta & k\\ 0 & 0 & 1\end{pmatrix}$$

in G_P, such that transforming first by g_P and then by f is the same as transforming first by f and then by g_Q

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Q\\ \downarrow{\scriptstyle g_P} & & \downarrow{\scriptstyle g_Q}\\ P & \xrightarrow{\ f\ } & Q\end{array}$$

Therefore any locus L_P in P yields an isomorphic locus $L_Q = f(L_P)$ in Q. Any property of L_P which is invariant under G_P is *a fortiori* invariant under the subgroup G_Q, and so yields a property of L_Q invariant under G_Q. Hence a P-theorem implies a Q-theorem. Combining this with the previous result we have: any projective theorem implies a euclidean theorem.

8. Non-uniqueness

The map f above is by no means unique, for we can choose coordinates $\{a, b, c\}$ in P with respect to an arbitrary triangle of reference and unit point. In fact it is possible to map any four chosen points of P to any four chosen points of Q, provided that no three of either four are collinear. Any such choice determines a unique map, and different choices in general determine different maps. Therefore the same P-locus (or P-theorem) may, under different choices, give rise to different Q-loci (or Q-theorems), and hence to different E-loci (or E-theorems). In particular, if we want to " project two given points X, Y of P into the circular points at infinity ", we choose the map f such that $f(X) = I$, $f(Y) = J$.

9. The reverse process

We have seen that any P-theorem implies one or more E-theorems. We now come to the crucial question: does an E-theorem imply a P-theorem? In practice the question often arises in the following form: given a P-theorem, does this follow from some E-theorem? In other words, can we prove a given projective theorem by euclidean methods? The logical objections to this procedure are summarised under the four headings, which we subsequently discuss in detail.

(i) Non-invariance.
(ii) Restrictions on the hypothesis.
(iii) Indeterminate loci.
(iv) Non-universality.

10. Non-invariance

Since G_P is larger than G_E, a property that is invariant under the smaller group G_E may not be invariant under the larger group G_P (for example metrical properties). Therefore if we want to deduce P-theorems, which by definition must be about P-invariants, from E-theorems, we have judiciously to choose only those E-theorems about properties which have projective meaning, and can be interpreted as invariants under the larger group (for example theorems about cross-ratios or linearity).

11. Restriction on the hypothesis

Consider a P-theorem whose hypothesis involves more than four arbitrary points (or some configuration with an equivalent degree of freedom). The cross-ratio subtended at one of the points by four of the others is in general a complex number which is not real. Therefore it is impossible to find an isomorphism $f: P \to Q$ such that the images of these five points are all real, because f preserves cross-ratios. Consequently, even if we had been able to prove the corresponding E-theorem word for word, we could not deduce the desired P-theorem from it, because we could not map the arbitrary hypothetical P-configuration into the real plane. In other words we could not choose a triangle of reference and unit point so that the arbitrary hypothetical P-configuration had real coordinates.

Desargues' theorem is a case in point, because the hypothesis demands two arbitrary triangles in perspective, and, given five arbitrary points, it is easy to choose them as five of the six vertices of two triangles in perspective. Therefore the projective Desargues' theorem cannot be deduced from the euclidean Desargues' theorem.

On the other hand the projective theorem about the polar line of a point with respect to a triangle can be readily proved by euclidean methods. The P-theorem states: if AM, BM, CM meet BC, CA, AB in D, E, F, respectively, and if EF, FD, DE meet BC, CA, AB in X, Y, Z, respectively, then X, Y, Z are collinear. The hypothesis involves exactly four arbitrary points A, B, C, M. The conclusion requires the collinearity of three points X, Y, Z which are uniquely constructed from the original four. It suffices only to prove the theorem in one special real case. For then we can map the four arbitrary points A, B, C, M of the general case onto the four points of the special case, and deduce the collinearity of X, Y, Z from that of their real images. For the special real case, take M to be the centre of the equilateral triangle ABC; then X, Y, Z are collinear on the line at infinity E^∞. To be quite precise, we have in fact deduced a P-theorem from an $(E \cup E^\infty)$-theorem in this case.

The conclusion of this section is that only projective theorems with a relatively simple hypothesis can be proved by euclidean methods.

12. Indeterminate loci

Suppose that the P-theorem, that we are hoping to prove by euclidean methods, is that a given construction in P determines a certain locus L_P. For example, consider the following theorem: if XY is a fixed chord of the conic S, then the locus of a point Z, which moves so that the tangents from Z to S divide ZX, ZY harmonically, is a conic touching S at X and Y.

Suppose then that we are given some such P-theorem, and suppose

Therefore any locus L_P in P yields an isomorphic locus $L_Q = f(L_P)$ in Q. Any property of L_P which is invariant under G_P is *a fortiori* invariant under the subgroup G_Q, and so yields a property of L_Q invariant under G_Q. Hence a P-theorem implies a Q-theorem. Combining this with the previous result we have: any projective theorem implies a euclidean theorem.

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Suppose then that we are given some such P-theorem, and suppose

also that we can project suitable points to the circular points at infinity (or in other words choose an isomorphism $f: P \to Q$), and can restrict the image construction in Q to a real construction in the subset E. Suppose further that we are able to determine by euclidean methods the resulting real euclidean locus L_E in E. Can we now deduce the larger locus L_Q in Q, and hence the desired locus L_P in P? This is what is meant by proving the original P-theorem by euclidean methods. The answer is in general no.

To be more specific, suppose that in fact the locus L_E is the unit circle

$$x^2+y^2=1.$$

We cannot then deduce that L_Q is a conic, without first proving that L_Q is *irreducible*, i.e. consists of only one component. For otherwise L_Q might have been the quartic curve, the pair of conics,

$$(x^2+y^2-1)(x^2+y^2+1)=0.$$

Of course the significant feature of the second of these two conics is that it contains no real points in Q. However, when we pass to the corresponding locus L_P in P, given by

$$(x^2+y^2-z^2)(x^2+y^2+z^2)=0,$$

this significant feature is lost, since it is no longer an invariant of the geometry. There is no qualitative feature to distinguish the character of the second conic from that of the first, because there is a projectivity in G_P transforming one onto the other.

Therefore from the fact that L_E is a circle, we can deduce only that L_P has one component that is a conic—it may have other components, which may be points, lines, conics, cubics etc., all of which are equally important parts of the locus.

On the other hand suppose that we do have the additional information that L_P is irreducible; then we can deduce that L_P is a conic, and is given by the equation $x^2+y^2=z^2$. However, this deduction itself depends upon the following deep algebraic theorem:

> *An irreducible complex algebraic variety, defined in complex euclidean n-dimensional space by equations with real coefficients, is uniquely determined by its real cross-section, provided that the real dimension of the cross-section equals the complex dimension of the variety.*

The theorem is proved by local analytic continuation. As a corollary to the theorem, we can deduce from the uniqueness, that the variety will have the same equations as its real cross-section.

The theorem is applicable in our case, because we have supposed that L_P is irreducible, and because the real dimension of the circle $x^2+y^2=1$ in the real euclidean plane, and the complex dimension

of the conic $x^2+y^2=1$ in the complex euclidean plane, are both equal to 1.

The proviso about dimension in the hypothesis of the theorem is shown to be necessary by the following example. The circle

$$x^2+y^2=1, \quad z=0, \quad t=0,$$

in real euclidean 4-dimensional space, is the real cross-section of each of the three irreducible varieties in complex euclidean 4-dimensional space

(i) the conic $x^2+y^2=1$, $z=0$, $t=0$;
(ii) the quartic 3-fold $(x^2+y^2-1)^2+z^2+t^2=0$;
(iii) the quartic 3-fold $(x^2+y^2-1)^2+z^4+t^4=0$.

It is easy to construct an infinite number of different irreducible varieties with this same cross-section, but only one of them, number (i) above, has complex dimension 1, equal to the real dimension of the original circle.

Having debated the general theory, let us now return to the projective plane, and enquire of the projective theorem given at the beginning of this section, whether it can be proved by euclidean methods. We are given that XY is a fixed chord of the conic S, and that Z moves so that the tangents from Z to S divide ZX, ZY harmonically. We have to show that the locus L_P of Z is a conic touching S at X, Y.

Let S' denote the circle

$$x^2+y^2=\tfrac{1}{2}$$

in Q. In detail, S' consists of all finite points (x, y) satisfying $x^2+y^2=\frac{1}{2}$, and all infinite points $\{x, y\}$ satisfying $x^2+y^2=0$. We can project S into S' by projecting X, Y into the circular points at infinity and by projecting two other points of S that separate X, Y harmonically into the ends of a real diameter of S'. Then Z projects into a point Z' such that the tangents from Z' to S' are perpendicular. Restricting our attention to the real cross-section, the locus L_E of Z' in E is the unit circle

$$x^2+y^2=1.$$

We can now appeal to the above theory, and deduce that the locus L_P of Z is a conic, which touches S at X, Y since the two circles are concentric, provided we verify that L_P is irreducible. Therein lies the rub, because the easiest way to prove that L_P is irreducible is to prove that it is a conic. In fact it is necessary to prove, by projective methods, the very theorem that we set out to prove by euclidean methods. Actually the locus of Z is none other than the harmonic locus of the tangential conic S and the point pair X, Y.

The conclusion of this section is that it is rare that euclidean methods alone will furnish a proof of a projective theorem if a locus is to be found.

13. Non-universality

The fourth objection to using euclidean methods of proof is the non-universality of such proofs for projective geometries over arbitrary fields. Most projective theorems are universally true over all algebraically closed (commutative) fields, and therefore projective proofs are desirable which appeal only to properties derived from the field axioms. Euclidean proofs, however, are liable to make use of notions of order, distance, angle, perpendicularity, and even continuity and connectedness, all of which are foreign to algebraic geometry proper. Of course such criticism is irrelevant if we confine ourselves to just the real and complex number fields.

14. Aesthetic considerations

The above four objections to euclidean proofs of projective theorems were hard facts on the correctness or incorrectness of the proofs; it is a prima facie requirement of any proof that it be correct. We conclude with a discussion of opinions and matters of taste.

It is sometimes claimed that it is no longer fashionable to use euclidean proofs; this is nonsense. Nevertheless, two factors are very important to a mathematician. Firstly a mathematical subject should be elegant, and a geometry should be all-of-a-piece, like any work of art. It gives aesthetic pleasure to prove projective theorems by projective methods, especially because of the universality mentioned in the last section.

Secondly, although each mathematical discipline should possess its own internal elegance, often the interconnection between two different disciplines is very fruitful. Whereas it is natural to prove euclidean theorems by euclidean methods, it is also interesting to deduce euclidean theorems from projective theorems, because this reveals which parts of euclidean geometry are inherited from the underlying linear structure, as opposed to those parts which depend essentially upon the euclidean structure itself. To go by the reverse process, on those few occasions when it is correct, can be amusing but it is not illuminating; for although it may be logically correct it affords little mathematical insight.

In support of those who relish the interconnection between different disciplines, sometimes there appears, in an otherwise self-contained discipline, a fundamental result which cannot be proved without the use of exotic methods; this is always mysterious. A striking case in point (which is nothing to do with the connection between the geometries that we have been discussing) is the essential

use of topological methods needed for the proof of the fundamental theorem of algebra, that the complex field is algebraically closed.

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CALENDAR REFORM

By W. F. Bushell

The difficulty of a good calendar lies in the fact that the earth travels round the sun in 365¼ days, or to be precise in 365 days 5 hours, 48 mins., 46 secs. which, we should note, is less than 365¼ days by just over 11 minutes.

Now one of the essentials of a good calendar is that the seasons should come on or about the same Calendar dates every year. It would create confusion if June gradually over the centuries became a winter month, which is true of the Mohammedan year which is based on the moon, as there are only 354 days in the year. Indeed 34 of their years approximately equal 33 of ours which depends on the sun. That is why Ramadan, their month of fasting, comes at different seasonal periods, and causes hardship when it comes in summer, as they only eat and drink when dusk has fallen. However I am not going to write of lunar years, but of years such as ours, dependent on the sun.

Our difficulty is that the seasonal year, or the time taken by the earth to go round the sun, is not an exact multiple of the day, the time taken by the earth to rotate on its axis. Of course the day and the seasonal year are astronomical periods and we cannot alter them, but other things such as the week, the month and the hour are merely man-made and artificial.

What, very briefly, is the history of our calendar? We might start with Julius Caesar, who is said to have become interested in calendar reform because he found himself in Northern Gaul putting his army into winter quarters in the midst of a blustering snow storm, and was amazed to find that the calendar marked spring instead of winter. Voltaire says "Roman generals always triumphed, but they never knew on what day they triumphed." Hence, when he became dictator in Rome he decided in 46 B.C. on the advice of the astronomer Sosigenes of Alexandria to take the length of the

seasonal year, i.e. the time taken by the earth to revolve round the sun to be 365¼ days, and decreed that each year should consist of 365 days, but that every fourth year should be 366, the so-called leap year. The calendar had got badly out of gear, and to effect this change and turn the old calendar into the new, the year we now call 46 B.C. or what the Romans called 708 A.U.C. (*ab urbe condita*), had to be extended to 445 days. This was always known as the year of Confusion. Actually the seasonal year is 11 minutes less than 365¼ days, and if 11 minutes is multiplied by 128 the result is 24 hours. This means that in 128 years the Julian calendar fell into error by one day.

For lovers of Arithmetic we might add that the Julian year is 365·25 days, whereas the real year is 365·2422 days, a difference of ·0078 days or 11 minutes.

There is a further point of interest. At the time of Caesar's reforms it was arranged that the vernal equinox, or the day when night and day were of equal length, and regarded as the beginning of spring, should be on March 25th, and it was one of the purposes of the reform that it should continue on that date, or thereabouts. Now the Council of Nicaea, which Constantine called soon after the Roman Empire had officially accepted Christianity, occurred in A.D. 325 which was about 370 years later, and by that time the equinox had retrogressed to March 21st, i.e. some 4 days. There are reasons into which I will not enter why it is retrogressed 4 days instead of 3 days, which would seem the correct figure as 128 goes into 369 nearly 3 times not 4 times.

This Julian calendar went on for 16 centuries until 1582 when Pope Gregory XIII, finding that by that time the vernal equinox had receded to March 11th, and remembering that the date of Easter had consequently retrogressed as well, because it depended on the vernal equinox, and realising that this gradual retrogression would continue, determined to put the vernal equinox back to March 21st and not March 25th as originally settled by Caesar, probably on the grounds that it had occurred on March 21st in A.D. 325 at the time of the first great Council of the church, the council of Nicaea. Hence he decreed that he would drop 10 days out of the calendar and in fact October 4th was followed by October 15th. He was advised by the Astronomer Clavius, the Lilio brothers and others.

I should perhaps add that the vernal equinox is really the moment when the sun gets to a particular point, i.e. the first point of Aries which is the intersection of the celestial equator with the ecliptic. Actually in 1953, it occurred at 10 p.m. on March 20th. That is why I should speak of March 21st as the approximate date. In 1954 it occurred at 4 a.m., in 1958 at 3 a.m. and in 1959 at 9 a.m., all on March 21st.

To avoid error occurring in future, Gregory further determined that in 400 years there should be 97 leap years and not 100. Thus a century year is not a leap year unless it is divisible by 400, i.e. 1900 was not a leap year, though 2000 is, and we have to bear this in mind when we say that any year divisible by 4 is a leap year. This Gregorian adjustment meant that the seasonal year, i.e. the time the earth takes to go round the sun is about 26 seconds only less than the *average* calendar year, and we go wrong by one day in 3323 years which scarcely matters. This was a clever adjustment.

This small alteration was only accepted by Roman Catholic countries. Indeed England did not adopt this calendar until 1752, but by that time it was necessary to drop 11 days not 10 out of the calendar and hence September 3rd of that year was returned as September 14th. This is said to have been followed by riots, and shouts of " Give us back our 11 days ", perhaps due to conservatism, or to the idea that the worker was being defrauded of his wages. But such unrest has been absurdly exaggerated. The change was really due to the pressure brought by the Earl of Chesterfield, famous for the letters to his son. He was helped by the Earl of Macclesfield, President of the Royal Society, and Bradley the Astronomer Royal.

It is interesting to note that Americans celebrate the birthday of George Washington on February 22nd though he was actually born on February 11th, according to the calendar *then* in existence. Further, when the Americans bought Alaska from Russia for 7,200,000 dollars in 1867, they had by that time to drop 12 days out of the Alaskan calendar, as the Russians still kept to the old style, and in fact did so until the Revolution of 1917.

The following dates illustrate how slowly the change permeated the world. The Gregorian calendar was adopted by

Japan in 1873
China in 1912
Turkey in 1916
Yugoslavia and Roumania in 1919 and
Greece in 1923.

This then is our calendar, initiated by Julius Caesar about 2000 years ago, and slightly modified by Pope Gregory nearly 400 years ago.

It might be interesting, before we pass on, briefly to consider the question of the countries that use a lunar year.

Many millions of Mohammedans still keep the lunar month of 29½ days. This means 354 days in the year. The months are alternately 29 and 30 days. Hence the ecclesiastical festivals rapidly change seasonally. The period of fasting called Ramadan, the name of the ninth month, may be in the summer or the winter.

They may not eat or drink during daylight, and strictly may not even swallow their own spittle. A summer Ramadan can therefore be very trying in a hot climate. In the large towns guns announce the beginning and end of night. I was in Marrakesh in Morocco on April 8th, 1958 and Ramadan ended at 3.29 a.m. when the orb of the New Moon became visible. It had commenced on March 11th.

Further, the Jews have a lunar month, but interpolate another month on occasions so that Passover remains in the Spring. All their festivals are tied to the moon. So indeed is our Easter.

We note 1 year = 365·2422 days
1 lunar month = 29·5306 days
But $365{\cdot}2422 \times 19 = 6939{\cdot}602$
and $29{\cdot}5306 \times 235 = 6939{\cdot}69$

Hence 235 lunar months approximately equal 19 years so that 19 years form a kind of Saros. During these nineteen years the Jews put in 7 extra months. They date their calendar from 3761 B.C. the so-called foundation of the world, and our year of 1959 would correspond to theirs of 5720, or the 19th year of the 301st cycle.

Now we come to our man-made months which can be arranged to suit our convenience apart from astronomical considerations. Julius Caesar arranged the length of the months admirably. He ordered that starting from January they should consist of 31 and 30 days alternately, i.e. January 31, February 30, March 31, and so on. This gives 366 days and hence he ordered that in a non-leap year, i.e. 3 years out of 4, February should only have 29. But his successor Augustus handled the matter badly. To commemorate his great uncle he ordered that the fifth month called Quintilis should be called July; the fifth because then the year started in March. He demanded also that the Sixth month Sextilis should be called August after himself. Unfortunately, however, this month August only had 30 days whereas July had 31. Hence to equalize the honour, he gave it 31 days taking one from February. This meant that July, August, September all had 31 days so that he took a day from September and gave it to October, and a day from November and gave it to December. Thus the vanity of Augustus has meant that for 2000 years we have all had to learn the ridiculous doggerel:

30 days have September
April, June and November
All the rest have 31
Except February alone
Which hath but 28 days clear.
And 29 in each leap year.

There are many varieties of this.

We might also notice that the names for the last 4 months of our year are now a misnomer. September, October, November and December mean the 7th, 8th, 9th and 10th month, which they originally were; this nomenclature is unsuited to our modern calendar, but does no real harm.

Let us now examine some obvious defects of our calendar.

(1) the first 6 months of the year contain 181 days, or 182 in a leap year. The second 6 months contain 184 days. So the four quarters contain 90, 91, 92, 92 days.

(2) the week-day name of each calendar date varies from year to year. Thus if Christmas Day falls this year on a Thursday it will, next year, fall on a Friday, or even on Saturday if a leap year intervenes.

(3) In the ecclesiastical calendar some holy days, and hence some civil life holidays, such as Christmas Day are observed as fixed calendar dates, such as Dec. 25th. Others such as Good Friday or Easter Sunday fall on fixed days of the week. But Easter Sunday falls on different calendar dates every year. Most of the ecclesiastical holy days are at fixed intervals before and after Easter which itself can fall on any date from March 22nd to April 25th, both inclusive, a range of 35 days. It actually fell on April 25th, the last possible date in 1943. In the four centuries from 1700 to 2100, Easter falls on April 25th on four occasions only, in 1734, 1886, 1943 and 2038. The earliest possible date is March 22nd and during these four centuries this only occurred twice, in 1761 and 1818.

We all know that it is the first Sunday after the first Paschal or hypothetical full moon after the Vernal equinox, as settled by the Council of Nicaea. This results in astonishing confusion; there is little doubt that if the calendar is reformed, the churches would in time agree to a fixed Easter, but I wish to make it clear that this is a separate question, and is a matter for the churches. A fixed Easter need have no connection with the reformed calendar I am about to suggest. Actually the Roman Catholic Church has stated there is no doctrinal difficulty about a fixed Easter, but are unwilling to act without a general council of the church. An act has gone through our parliament fixing Easter in the second week of April, as soon as the various churches agree, and, as you will see, this would always be April 8th or April 15th under the new calendar I am about to describe.

The two more important ways in which our calendar could be improved are

(1) the first 6 months should have the same number of days as the second 6, and all the quarters should be equal.

(2) Calendar dates should have the same week day name every year, i.e. January 4th must always be, (say), a Wednesday, and April 8th always a Sunday.

Now these points are satisfied by the so-called World Calendar which is the only reformed calendar with any chance of acceptance. It has been approved by a considerable number of Governments. It has been on the agenda of the United Nations Organization and will surely be one day accepted. It was first suggested by a Roman Catholic priest, Mastrofini in 1834, and its essential feature is a 364 day year, with another added outside the week. 364 days means of course exactly 52 weeks. Thus if January 1st is a Sunday, so always January 1st will be a Sunday, and the calendar is perpetual. Here are its leading characteristics;

(1) There will be 12 months named as now. The old suggestion of 13 months of 28 days each has been abandoned. We notice that 13 times 28 is 364, which, with one day outside the week, would effect the same purpose.

(2) There will be 4 quarters each of 91 days. The first month will always have 31 days, the 2 other months only 30. Thus each half year will contain 182 days or the half of 364.

(3) The Calendar will retain the present Leap Year rule, but in each year there will be one day unaccounted for, and in Leap Year two. Hence in each year there will be a day after December 30th to be called Year End World Holiday, and in each Leap Year yet another day after June 30th to be called Leap Year day. It is suggested that both these days should be world holidays.

(4) Most important of all in this calendar, all dates will have the same week day name. If, as is suggested, January 1st is a Sunday, so also will be the 1st day of April, July and October. Christmas Day will always be a Monday and, as an example, July 3rd will always be a Tuesday and so on. This is the most spectacular change.

(5) We note also that 13 days of each quarter will be Sundays and 78 always weekdays. Indeed by putting January 1st as a Sunday, each month will have 26 week days and 4 or 5 Sundays. This should be of considerable advantage in business.

(6) If this could have been started in 1961 when, in the Gregorian calendar January 1st was a Sunday, there would have been the minimum dislocation in civil and religious life.

It is worth remembering that Trygve Lie, the late Secretary General, gave the United Nations an unbiased summary of the advantages of this new calendar. He said "All years would be identical except Leap Year with its supplementary day. There are four equal and identical quarters, very convenient for certain aspects of everyday life.

Statistical survey, budgetary estimates, financial operations and economic and social plans can be drawn up in much simpler fashion.

Periodic events such as the convening of Parliament can be permanently fixed as regards date and week day. The stability of the

calendar makes it possible to contemplate the stabilization of festivals which are at present moveable. It would help to improve the measurement of time from the economic and social point of view."

In this short article I cannot discuss how New Year's Day has varied in the course of history but there is a point of some interest I might mention in passing. We dropped out the 11 days in 1752. The year had legally commenced, up to that time, on March 25th or lady-day, i.e. the feast of the Annunciation. It was then enacted it should commence on January 1st as we accepted the Gregorian adjustment. The national accounts had been made up to March 25th the end of the year, but in order that they might still reflect 365 days, this was altered to April 5th. That is why, in our income tax returns, we still regard one year as extending from April 5th to April 5th, as all tax payers know, though many have wondered why so strange a date has been chosen. It is an interesting reminder of the old calendar.

It is a platitude to say that nations are being drawn more closely together. We want order, stability, harmony and co-ordination. The calendar is indispensable, and it does seem ridiculous that we should still be using one initiated by Julius Caesar 2000 years ago, with a slight Gregorian modification in 1582. I know very well that changes are difficult. The Julian reforms came after years of agitation. Indeed that year, 46 B.C., was called the year of confusion rather than the year of good riddance of bad rubbish. So it was with the Gregorian reform which had been suggested as desirable by Roger Bacon 300 years before it was implemented. It was often discussed, and even Copernicus in 1514 was asked his opinion, and it was not until the Pope and his advisers realised that ultimately Easter would recede steadily to the winter that they agreed to action.

In the same way the Standard hour system took many years to bring about. People have forgotten that it was not till 1883 that Sir Sandford Fleming, a Canadian, persuaded the American continent to adopt Standard time, whereby the earth was divided into 24 zones by meridians 15 degrees of longitude apart, starting from the Greenwich meridian, with the necessary corollary of the international date line in the middle of the Pacific. Incidentally it is a matter of national pride that the world recognises the Greenwich meridian as one from which all time is measured, and I always feel that this is to some extent due to the reputation of the 19th-century British astronomers. It was also partly due to the fact that the association of Greenwich with regard to external time keeping first arose in regard to navigation. The British Nautical Almanack, started in 1767, emanated from Greenwich and was used by nearly all ships for longitude purposes. Hence it was natural to use the

Greenwich meridian. Actually, Greenwich mean time was legally made universal in Great Britain in 1880.

Further, an International Conference met at Washington in October 1884 and it was resolved by 21 votes to 1, with France and Brazil abstaining, to approve the American proposal that the Greenwich meridian should be adopted, largely on the grounds that it would be, under the circumstances, the most generally convenient plan. This agreement gradually became effective, though there does not seem to have been any formal ratification by various governments. I might add that the Airy Transit Circle at Greenwich has, for purpose of time keeping, found the time on 667,000 occasions, from 1851–1951, though I understand its life is now over. Indeed it is an historic monument whose centenary was almost unknown to the public.

It is inertia with which we have to contend; people do not take kindly to new ideas. The suggestion of the new calendar first came in 1834, and only in recent years has been fully considered. There is no reason why in this modern age we should use an antiquated device, hoary with age, and out of keeping with the precision demanded to-day.

Let me sum up the advantages of the new Calendar:

(1) Days and dates agree from year to year.
(2) Each month has 26 week days.
(3) Each month has the same day arrangements every year, and begins the same day, i.e. Sunday—Wednesday—Friday.
(4) Each quarter has 3 months of 91 days and 13 weeks.
(5) Each quarter begins with a Sunday. Each quarter ends with a Saturday.
(6) Holidays and anniversaries are stabilised on their regular days and dates.

Finally, it was Hamlet who said:

> "The time is out of joint; oh cursed spite,
> That ever I was born to set it right."

That is a challenge; we might well rewrite and declare:

> "Our time is out of joint; but oh delight
> That we are born this age to set it right."

W.F.B.

2942. An example of tripolar root-squaring

In 1936 I gave (*Gazette*, XX, 178) in some detail a solution of the equation

$$z^2-(2+i)z+(4+3i)=0, \qquad (1)$$

to illustrate the three stages in the determination of a root:

(i) Graeffe's root-squaring process for finding an approximation r to a modulus possible for a root, and approximations, for some known value of n, to $\cos 2^n\theta$ and $\sin 2^n\theta$, where θ is an angle to be associated with r;

(ii) Goursat's use of r and $\cot \frac{1}{2}\theta$ to identify θ from $\cos 2^n\theta$ and $\sin 2^n\theta$;

(iii) The construction of iterative formulae for improving indefinitely the values of x and y to which $r\cos\theta$ and $r\sin\theta$ as found in (i) and (ii) provide first approximations.

The problem having been brought again to my attention recently, I propose to use the same equation (1) to illustrate a solution which avoids all reference to the angle θ, and employs root-squaring only to find moduli, or rather the squares of moduli, which are called the *norms* of the roots; this application of root-squaring is due in essence to Apollonius.

In theory Graeffe's process makes no distinction between real coefficients and complex coefficients, and in the paper referred to I operated directly on (1), converting an equation

$$z_n^2-p_nz_n+q_n=0$$

with p_n, q_n complex into its successor by the recurrences

$$p_{n+1}=p_n^2-2q_n, \qquad q_{n+1}=q_n^2.$$

The simplicity is deceptive, for the number of arithmetical multiplications involved is no smaller than in turning an equation

$$z_n^4-b_nz_n^3+c_nz_n^2-e_nz_n+f_n=0$$

with real coefficients into its successor by the recurrences

$$b_{n+1}=b_n^2-2c_n,\ c_{n+1}=c_n^2-2b_ne_n+2f_n,\ e_{n+1}=e_n^2-2c_nf_n,\ f_{n+1}=f_n^2.$$

In the latter transformation the arithmetical routine is simpler and mistakes in sign are less likely; the norms are given directly, and separation of norms is more easily recognised than separation of the roots themselves.

The roots of (1) are two of the roots of

$$z^4-4+13-22+25=0, \qquad (2)$$

and writing $z_1=z^2$, $z_2=z_1^2$, and so on, we have from (2)

$$z_8^2-4{\cdot}090.10^{12}+4{\cdot}198.10^{24}+9{\cdot}233.10^{32}+5{\cdot}421.10^{44}=0,$$

an equation which separates approximately into the two equations

$$z_5^2 - 4{\cdot}090.10^{12} z_5 + 4{\cdot}198.10^{24} = 0,$$
$$4{\cdot}198 z_5^2 + 9{\cdot}233.10^{9} z_5 + 5{\cdot}421.10^{20} = 0.$$

In the equation for z_5, the coefficient of z_5^2 is $1{\cdot}770.10^{49}$; further squaring will not improve the approximation unless more significant figures are retained, and we can say that (2) has one pair of roots with a common norm α_1 given by $\alpha_1^{24} = 1{\cdot}770.10^{49}$, and one pair whose common norm α_2 is given directly by $\alpha_2^{24} = (5{\cdot}421^2/1{\cdot}770)10^{39}$ or more simply by $\alpha_2 = 25/\alpha_1$. To three figures, $\alpha_1 = 5{\cdot}88$, $\alpha_2 = 4{\cdot}25$.

The values of α_1 and α_2 show that the distances of the roots of (1) from the origin are between 2 and 2·5, and, I propose now to take concurrently the points $z = -2$ and $z = 2$ for fresh origins, writing $z = u - 2 = v + 2$ and denoting the norms of u and v by β and γ. For any position of z,

$$\beta + \gamma = 2\alpha + 8, \qquad \beta - \gamma = 8x.$$

If z satisfies (1), then u satisfies

$$u^4 - 12 + 61 - 154 + 169 = 0,$$

and v satisfies

$$v^4 + 4 + 13 + 14 + 17 = 0.$$

The equation in u has two norms β', β'' given approximately by $(\beta')^{12} = 1{\cdot}5400\,.10^{148}$, $\beta'' = 169/\beta'$, that is, by $\beta' = 14{\cdot}378$, $\beta'' = 11{\cdot}754$, and the equation in v has two norms γ', γ'' given approximately by $(\gamma')^{12} = 8{\cdot}2380\,.10^{28}$, $\gamma'' = 17/\gamma'$, that is, by $\gamma' = 8{\cdot}010$, $\gamma'' = 2{\cdot}122$.

If β_1, γ_1 are the u and v norms associated with α_1, and β_2, γ_2 the u and v norms associated with α_2, then $(\beta_1 + \gamma_1) - (\beta_2 + \gamma_2) = 2(\alpha_1 - \alpha_2)$ and therefore $(\beta_1 - \beta_2) + (\gamma_1 - \gamma_2) = 3{\cdot}26$. But

$$\beta_1 - \beta_2 = \pm(\beta' - \beta'') = \pm 2{\cdot}624, \quad \gamma_1 - \gamma_2 = \pm(\gamma' - \gamma'') = \pm 5{\cdot}888,$$

and of the four numbers $\pm 2{\cdot}624 \pm 5{\cdot}888$ the only one which is even approximately 3·26 is $-2{\cdot}624 + 5{\cdot}888$. Hence the grouping is beyond doubt, and we have $\beta_1 = \beta''$, $\gamma_1 = \gamma'$, $\beta_2 = \beta'$, $\gamma_2 = \gamma''$, and further $\alpha_1 = 5{\cdot}882$, $x_1 = 0{\cdot}468$; $\alpha_2 = 4{\cdot}250$, $x_2 = 1{\cdot}532$.

From this point the solution follows a familiar track. The equation in z is replaced by the pair of equations

$$f(x, y) \equiv x^2 - y^2 - 2x + y + 4 = 0, \quad g(x, y) \equiv 2xy - x - 2y + 3 = 0, \qquad (3)$$

with known approximations to x and y^2. From both equations it is obvious that y_1 is positive and y_2 negative, and the problem that remains is to improve on the two solutions (a, b) and (c, d) of (3), knowing that first approximations to these solutions are

$$a_1 = 0{\cdot}468, \quad b_1 = 2{\cdot}380; \quad c_1 = 1{\cdot}532, \quad d_1 = -1{\cdot}379.$$

The values of the partial derivates of f and g near (a, b) are approximately $f_x = g_y = -16/15$, $f_y = -g_x = -15/4$, and the pair of equations

$$f-(16/15)h-(15/4)k=0, \qquad g+(15/4)h-(16/15)k=0$$

is almost equivalent to

$$h=\tfrac{1}{15}f-\tfrac{1}{4}g, \qquad k=\tfrac{1}{4}f+\tfrac{1}{15}g.$$

Thus we may approach (a, b) by the iteration

$$a_{n+1}=a_n+\tfrac{1}{15}f(a_n, b_n)-\tfrac{1}{4}g(a_n, b_n),$$
$$b_{n+1}=b_n+\tfrac{1}{4}f(a_n, b_n)+\tfrac{1}{15}g(a_n, b_n).$$

In this example, the values of the partial derivatives near (c, d) are the negatives of the values near (a, b), and we may approach (c, d) by the iteration

$$c_{n+1}=c_n-\tfrac{1}{15}f(c_n, d_n)+\tfrac{1}{4}g(c_n, d_n),$$
$$c_{n+1}=c_n-\tfrac{1}{4}f(c_n, d_n)-\tfrac{1}{15}g(c_n, d_n).$$

From the original equation we have accurately $a+c=2$ and $b+d=1$, but it is perhaps better to reserve these relations as a final check than to dispense altogether with the independent evaluation of c and d. To eight places, as found before,

$$a=0{\cdot}46798\,290, \qquad b=2{\cdot}37963\,885.$$

(*received in February* 1960) E. H. NEVILLE

2943. The year 1961

1. $19^2+19^2=5^2+16^2+21^2$
   $19^4+19^4=5^4+16^4+21^4$ where $5+16=21$.

2. $61^2+61^2=9^2+56^2+65^2$
   $61^4+61^4=9^4+56^4+65^4$ where $9+56=65$.

3. $1961^2+1961^2=371^2+1749^2+2120^2$
   $1961^4+1961^4=371^4+1749^4+2120^4$ where $371+1749=2120$.
   We remark that the identity
   $A^2+A^2=B^2+C^2+D^2$
   $A^4+A^4=B^4+C^4+D^4$ where $B+C=D$,
   is only possible, if A has the Form m^2+mp+p^2.

4. $19+19+(20^2-19)+(20^2-19)$
   $=1^2+19^2+(20^2-19^2)+(20^2-1^2)$
   $19^2+19^2+(20^2-19)^2+(20^2-19)^2$
   $=1^4+19^4+(20^2-19^2)^2+(20^2-1^2)^2$
   $19^3+19^3+(20^2-19)^3+(20^2-19)^3$
   $=1^6+19^6+(20^2-19^2)^3+(20^2-1^2)^3$

5. $61+61+(62^2-61)+(62^2-61)$
   $=1^2+61^2+(62^2-61^2)+(62^2-1^2)$
   $61^2+61^2+(62^2-61)^2+(62^2-61)^2$
   $=1^4+61^4+(62^2-61^2)^2+(62^2-1^2)^2$

$$61^3+61^3+(62^2-61)^3+(62^2-61)^3$$
$$=1^6+61^6+(62^2-61^2)^3+(62^2-1^2)^3$$

6. $2\,.\,1961+2\,.\,[(1961+1)^2-1961]$
$$=1^2+1961^2+[(1961+1)^2-1961^2]+[(1961+1)^2-1^2]$$
$$2\,.\,1961^3+2\,.\,[(1961+1)^2-1961]^3$$
$$=1^6+1961^6+[(1961+1)^2-1961^2]^3+[(1961+1)^2-1^2]^3$$
$$2\,.\,1961^3+2\,.\,[(1961+1)^2-1961]^3$$
$$=1^6+1961^6+[(1961+1)^2-1961^2]^3+[(1961+1)^2-1^2]^3$$

7. $(-1961)^3+(-2\,.\,1961)^3+(3\,.\,1961)^3$
$$=(6\,.\,1961^3)+(6\,.\,1961^3)+(6\,.\,1961^3)$$
$$(-1961)^3\,.\,(-2\,.\,1961)^3\,.\,(3\,.\,1961)^3$$
$$=(6\,.\,1961^3)\,.\,(6\,.\,1961^3)\,.\,(6\,.\,1961^3).$$

Furthermore

8. $(-K)^3+(-2K)^3+(3K)^3+3\,.\,(12\,.\,K^3)$
$$=3\,.\,(6\,.\,K^3)+(-9\,.\,K^3)+(19\,.\,K^3)+(26\,.\,K^3)$$
$$(-K)^6+(-2K)^6+(3K)^6+3\,.\,(12\,.\,K^3)^2$$
$$=3\,.\,(6\,.\,K^3)^2+(-9\,.\,K^3)^2+(19\,.\,K^3)^2+(26\,.\,K^3)^2$$
$$(-K)^9+(-2K)^9+(3K)^9+3\,.\,(12\,.\,K^3)^3$$
$$=3\,.\,(6\,.\,K^3)^3+(-9\,.\,K^3)^3+(19\,.\,K^3)^3+(26\,.\,K^3)^3$$
$$(-K)^{12}+(-2K)^{12}+(3K)^{12}+3\,.\,(12\,.\,K^3)^4$$
$$=3\,.\,(6\,.\,K^3)^4+(-9\,.\,K^3)^4+(19\,.\,K^3)^4+(26\,.\,K^3)^4.$$

Take $K=1961$, then

9. $(10\,.\,1961^3)+(17\,.\,1961^3)+(-18\,.\,1961^3)\quad=3^3\,.\,1961^3$
$(10^3\,.\,1961^3)+(17^3\,.\,1961^3)+(-18^3\,.\,1961^3)=3^4\,.\,1961^3$.

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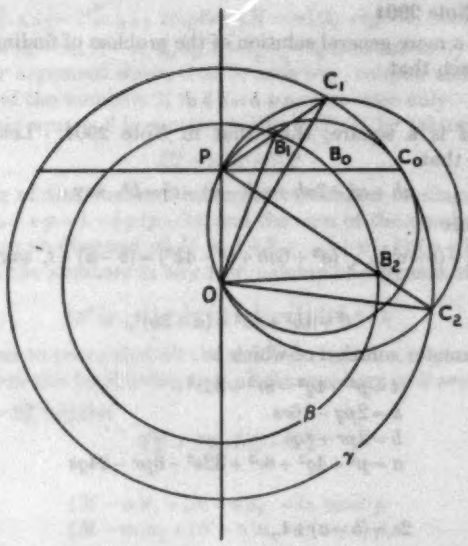
2944. On Note 2906.

1. β, γ are two concentric circles with P a point inside them. A half line through P meets β, γ at B, C respectively. It is required to show that BC is greatest when PBC is at right angles to the diameter OP.

A technique devised by Newton suggests a neat solution.

Let PB_0C_0 be the half line through P which gives the maximum length for BC. Then assuming that the length of BC is a continuous function there will be in the neighbourhood of B_0C_0 two positions B_1C_1, B_2C_2 on opposite sides of B_0C_0 such that $B_1C_1=B_2C_2$.

The triangles OB_1C_1, OB_2C_2 are congruent, whence angle C_1 is equal to angle C_2. O, P, C_1, C_2 are therefore concyclic. Now let C_1, C_2 converge on C_0. The limiting position of the circle OPC_1C_2



will be a circle OPC_0 touching γ at C_0. OC_0 is a diameter of this circle, whence angle OPC_0 is a right angle, and the result is established.

GEORGE TYSON

2. I give an alternative proof of the result that, if P is a fixed point within the inner of two concentric circles, and A, B, P, C and D are collinear points in order so that AD is a chord of the larger and BC a chord of the smaller, then CD takes its greatest value when AD is perpendicular to the diameter through P.

The power with respect to the outer circle of all points on the circumference of the inner circle is the same.

i.e. $$\begin{aligned} AC.CD &= \text{constant.} \\ &= (AB+BC)\,.\,CD \\ &= (CD+BC)\,.\,CD \\ &= CD^2 + BC\,.\,CD \end{aligned}$$

so that when CD^2 is a maximum, $BC\,.\,CD$ is a minimum.

i.e. when CD is a maximum, BC is a minimum,

and this occurs when the chord BC is at its greatest distance from the centre, viz. when BC is perpendicular to the diameter through P.

Jordanhill College of Education, Glasgow A. G. SILLITTO

Editorial Note. A trigonometrical solution was received from Professor H. Simpson.

2945. On Note 2904

We give a more general solution of the problem of finding integers a, b, c, d such that

$$a+b=c+d$$

and $ab+cd$ is a square, than that in Note 2904. Let $c=a+e$, $d=b-e$ so that

$$ab+cd=2ab+e(b-a)-e^2=k^2, \text{ say,}$$

and therefore

$$2e=(b-a)\pm\sqrt{(a^2+6ab+b^2-4k^2)}=(b-a)\pm t, \text{ say,}$$

whence

$$t^2+4k^2+8b^2=(a+3b)^2,$$

a four parameter solution of which is

$$\begin{aligned} t&=p^2-4q^2-8r^2+32s^2\\ k&=2pq-16rs\\ b&=2pr+8qs\\ a&=p^2+4q^2+8r^2+32s^2-6pr-24qs \end{aligned}$$

and

$$2e=(b-a)\pm t.$$

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2946. Magic squares of prime order

The arithmetic of remainders *modulo* a prime p provides a very simple method of constructing magic squares of prime order. We start by constructing two auxiliary squares T and U, denoting by T_n^m, U_n^m the numbers placed in the mth row and nth column of squares T, U respectively. The values of T_n^m are the residues $0, 1, 2, \ldots, p-1$, mod p, and the values of V_n^m are the residues $1, 2, 3, \ldots, p$ mod, p. T_1^1, and U_1^1 are chosen arbitrarily and t_1, t_2, u_1, u_2 are numbers from 1 to p subject to the following conditions mod p.

$$t_1-t_2\neq 0,\quad t_1+t_2\neq 0,\quad u_1-u_2\neq 0,\quad u_1+u_2\neq 0,\quad t_1u_2-t_2u_1\neq 0.$$

We define

$$\begin{aligned} T_n^m&=T_1^1+(m-1)t_1+(n-1)t_2\\ U_n^m&=U_1^1+(m-1)u_1+(n-1)u_2 \end{aligned}$$

We observe first that in each row, column and diagonal of T each of the numbers $0, 1, 2, \ldots, p-1$ occurs once only. For

$$\begin{aligned} T_n^m=T_N^m &\text{ implies } (N-n)t_2=0, \text{ mod } p,\\ T_n^m=T_n^M &\text{ implies } (M-m)t_1=0, \text{ mod } p,\\ T_m^m=T_M^M &\text{ implies } (M-m)(t_1+t_2)=0, \text{ mod } p \end{aligned}$$

and $T^m_{p-m+1} = T^M_{p-M+1}$ implies $(M-m)(t_1-t_2) = 0$, mod p, but none of $M-m$, $N-n$, t_1, t_2, t_1-t_2, t_1+t_2 is divisible by p.

A similar argument shows that in each row, column and diagonal of U each of the numbers $1, 2, 3, \ldots, p$ occurs once only.

The magic square S is constructed from T, U by taking

$$S^m_n = pT^m_n + U^m_n$$

The sum of the numbers in any row, column or diagonal of T is $1+2+\ldots+p-1 = \frac{1}{2}p(p-1)$, and the sum of the numbers in any row, column or diagonal of U is $1+2+\ldots+p = \frac{1}{2}p(p+1)$ so that the sum of the numbers in any row, column or diagonal of S is

$$\tfrac{1}{2}p^2(p-1) + \tfrac{1}{2}p(p+1) = \tfrac{1}{2}p(p^2+1).$$

It remains to prove that all the numbers $1, 2, 3, \ldots, p^2$ occur in S, and we prove this by showing that all the numbers in S are different.

For $S^m_n = S^M_N$ implies

$$T^m_n = T^M_N,\ U^m_n = U^M_N$$

that is

$$\begin{aligned}(M-m)t_1 + (N-n)t_2 &= 0, \text{ mod } p,\\ (M-m)u_1 + (N-n)u_2 &= 0, \text{ mod } p,\end{aligned}$$

and since $M-m$, $N-n$ are not both zero, therefore

$$t_1u_2 - t_2u_1 = 0, \text{ mod } p$$

and such values of t, u are excluded by the initial conditions.

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2947. On Note 2909

Mr. R. H. Macmillan's second formula for square root is a particular case of Barrow's Formula (1); to wit

If r be an approximation to $R^{1/n}$, where n is not less than unity a closer approximation, r', is given by

$$\frac{r-r'}{r+r'} = \frac{1}{n}\cdot\frac{r^n-R}{r^n+R} \tag{1}$$

or, put otherwise

$$\frac{r'}{r} = \frac{(n-1)r^n + (n+1)R}{(n+1)r^n + (n-1)R} \tag{2}$$

This gives cubic convergence.

Perhaps only survivors from the days when desk calculators were instruments for craftsmen, rather than bludgeons for the unskilled, are now aware of the enormous bulk of literature on high precision formula for fractional powers. However, increased higher storage

speeds in automatic computers is leading to rediscovery of that forgotten computational operation, division. In particular, rational polynomial fractions approximating closely to a function over a given range may sometimes be more convenient than more widely applicable recursive methods.

Much labour and talent will be wasted if the present revival of interest should lead to repetition of results rather than their improvement. This Gazette, alone, contains a wealth of material. For fractional powers, I would commend for instance the contributions of Birch (2), Richmond (3), Lowry (4), and Bickley (5).

An entertaining and possibly useful variation on the high precision formula is the rough approximation of extreme simplicity, over a given range. I submit, as initial provocation, the following

$$2^{1/n} \simeq \frac{35n+12}{35n-12} \qquad 1 \leqslant n$$

$$\log_2 a \simeq \frac{35}{12} \cdot \frac{a-1}{a+1} \qquad 1 \leqslant a \leqslant 2$$

or, better,

$$\simeq 3\frac{a-1}{a+1} - 0{\cdot}0076$$

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3. H. W. Richmond, "Cube Root", *Math. Gazette* **28**, 278, pp. 20–21 (Feb. 1944).
4. H. V. Lowry, "Approximation to the Roots of an Equation", *Math. Gazette*, **29**, 287, pp. 233–234 (Dec. 1945).
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2948. On Note 2909

Note 2909 can obviously be extended to give better approximations than formulae (1), (2) and (3), though they would soon become too complicated to be useful. The following, which might be called the fifth-order approximation, is, however, quite simple to use, and gives extremely accurate results with quite a crude initial value.

In the notation of Note 2909, if r is an approximation to the square root $r+h$ of a, then since

$$(r+h-r)^5 = (r+h)^5 - 5(r+h)^4 r + 10(r+h)^3 r^2 - 10(r+h)^2 r^3 + 5(r+h)r^4 - r^5,$$

$$h^5 = (r+h)(a^2+10ar^2+5r^4) - r(5a^2+10ar^2+r^4)$$

Hence $$r+h = \frac{r(5a^2+10ar^2+r^4)+h^5}{a^2+10ar^2+5r^4}$$

Neglecting h^5, we thus have

$$r+h \approx \frac{r(5a^2+10ar^2+r^4)}{a^2+10ar^2+5r^4} \qquad (1)$$

or $$h \approx \frac{4r(a^2-r^4)}{a^2+10ar^2+5r^4} = \frac{4r(a+r^2)(a-r^2)}{a^2+10ar^2+5r^4} \qquad (2)$$

e.g. If $a=10$, $r=3$,

$$h \approx \frac{4\,.\,3\,.\,19\,.\,1}{1405} = \frac{228}{1405} = \cdot 16227758$$

giving $r+h = 3\cdot 16227758$ (cf. $\sqrt{10} = 3\cdot 162277660\ldots$)
[The error in r is about 5%, while the error in $r+h$ is only about 1 part in 40 million!]

Police Station, Cainscross, Stroud, Glos. W. J. GODDEY

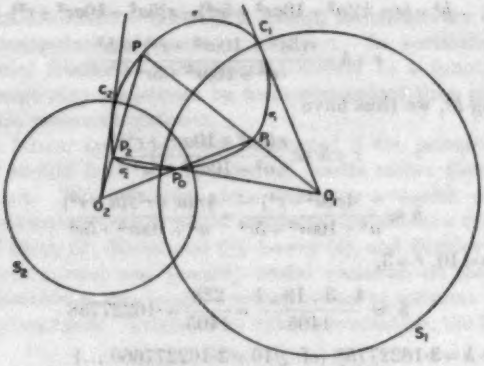
2949. Self inverse curves

On page 280 of *Theoretical and Practical Geometry* by the Irish Christian Brothers (published by M. H. Gill, Dublin, about 1908) will be found a " proof " of the theorem that the inverse of a circle with respect to a point not on the curve is a circle. Without reproducing the details of the proof here, it is correctly shown that the inverses with respect to the centre of inversion (supposed outside the given circle) of the four points which are the intersections of any two secants through the centre of inversion with the given circle, are four concyclic points. The inverse is then a closed curve such that any two secants through a particular point (the centre of inversion) meet it in four *concyclic* points. In other words, it is self-inverse with respect to *one* point for a suitable radius of inversion. From this it is concluded that the inverse curve is a circle.

Comment: *Non sequitur.*

The following is a method of constructing a closed curve (which need not be a circle) which is self-inverse even with respect to *two* points, any secant through either of which meets the curve in two points at most:

Let S_1, S_2 be two orthogonally related circles with centres O_1, O_2 respectively, and σ be a smooth flexless arc outside them but meeting them orthogonally at C_1, C_2. σ should be such that a ray through



O_1 or O_2 meets it in one point at most. σ_1 is the inverse of σ with respect to S_1 and σ_2 is the inverse of σ with respect to S_2. Clearly σ_{12}, the inverse of σ_1 with respect to S_2, is the same as σ_{21}, the inverse of σ_2 with respect to S_1: for if P be any point on σ then its inverse P_1 with respect to S_1 will be on σ_1 and its inverse P_2 with respect to S_2 will lie on σ_2. Let P_0 be the inverse of P_1 with respect to S_2. If two points are inversely related to a circle, this relationship is unaltered by inversion. Also S_1 is self-inverse with respect to S_2 (to which it is orthogonal).

Hence the circle S_1, P and P_1 invert with respect to S_2 into circle S_1, P_2 and P_0 respectively, where P_0 is the inverse of P_2 with respect to S_1. Hence P_0 is obtained either by inverting P with respect to S_1 and the resulting inverse point with respect to S_2 or by inverting P with respect to S_2 and the resulting inverse point with respect to S_1. Therefore $\sigma_{12} = \sigma_{21}$. Since σ meets S_1, S_2 orthogonally and the angle between two lines is unaltered by inversion, the arcs σ, σ_1, σ_2, σ_{12} or σ_{21} together form a rounded closed curve which is self inverse with respect to the points O_1, O_2 (for suitable radii of inversion) a secant through either of which meets it in two points at most.

If the resulting closed curve is to be *flexless*, O_1 and O_2 should lie outside every circle of curvature of the original arc σ.

The closed branch of a circular cubic affords an instance of the type of curve whose construction is shown above. This branch is self-inverse with respect to the two points on the other branch at which the tangents are parallel to the real asymptote.

It is not necessary that the initial arc σ should cut each circle orthogonally or that it should be convex; but in this more general case, the derived curve, though closed, will not be a rounded curve.

Kingston Grammar School H. G. WOYDA

2950. A property of two conics associated with Poncelet's theorem

1. Poncelet's Theorem states that if two conics S and S' satisfy the condition that there is a polygon* of m vertices such that its vertices lie on S and the lines joining adjacent vertices touch S' then there are infinitely many such polygons. It may be shown by similar methods (that is, by the use of a symmetrical (2, 2) correspondence on S) that if S and S' satisfy the condition that there is a polygon of n vertices such that its vertices lie on S and adjacent vertices are conjugate with respect to S' then there are infinitely many such polygons.

It is a rather surprising result that the first condition for $m=4$ implies the second condition for $n=6$ and conversely.

2. There are several ways of proving this result.

(i) It may be shown that, in the usual invariant notation, the condition is in each case

$$\Theta'^2 - 4\Delta'(\Theta\Theta' - \Delta\Delta') = 0.$$

However, this algebraic proof gives no insight into the geometry.

(ii) Let the first condition be satisfied for $m=4$. If A is one of the points of intersection of S and S', and the tangent to S' at A meets S again at X, then three adjacent vertices of the quadrangle containing A are XAX. If the other vertex is B, then B must be another common point of S and S', and the tangent to S' at B must pass through X. (It follows that the tangents to S' at the other common points meet on S.) Hence $XAXB$ is a (degenerate) quadrangle satisfying the first condition. However $XAAXBB$ is a (degenerate) hexagon satisfying the second condition. The converse result clearly follows.

(iii) The following proof is probably the most illuminating, because it shows how a hexagon can be derived from a given quadrangle, and vice versa.

We suppose a hexagon $AB'CA'BC'$ given, which satisfies the second condition. Since ABC and $A'B'C'$ are reciprocal triangles for S', AA', BB', CC' meet at O, say. We define the following points:

X, the intersection of BC and $B'C'$.
P, P', the points where the tangents from A to S' meet S again.
Z, Z', the points where these tangents touch S'.
Y, the intersection of AB and $B'C'$.

Now $(B'C', PP')$ on $S = (B'C', ZZ') = A(CB, PP')$, by taking polars with respect to S', $= (BC, P'P)$ on S. Hence B, B'; C, C'; P, P' are pairs of an involution on S, so PP' passes through O. Also,

* Whenever a polygon is mentioned, it is assumed that the vertices are considered in a certain cyclic order: the term "adjacent vertices" implies that the vertices are adjacent when considered in this order.

(BB', PP') on $S = (YB', ZZ') = A(C'C, PP') = (CC', PP')$ on S. Hence B, C; B', C'; P, P' are pairs of an involution on S, so PP' passes through X. It follows that P, P' are the points where OX meets S. Similarly, the tangents from A' to S meet S at P, P'. Hence $APA'P'$ is a quadrangle which satisfies the first condition.

The converse may be proved, though it is a little more difficult. We suppose a quadrangle $APA'P'$ given, which satisfies the first condition. We define the following points:

Z, Z', the points where AP, AP' touch S'.
B', C', the points where ZZ' meets S.
B, C, the points where the polars of C', B' for S' meet S (apart from A).
X, X', the intersections of BC, $B'C'$; BC', $B'C$.
O, the intersection of AA', PP'.
O', the harmonic conjugate of O with respect to A, A'.

It is a well-known result that the polars of O for S and for S' are the same line (which contains X, X', O'). We may prove, exactly as before, that PP' passes through X. Since AA' and PP' are conjugate lines for S', and the polar of A passes through X, it follows that X is the pole of AA' for S'. We may also prove, exactly as before, that BB', CC', PP' are concurrent, and since O and X are conjugate for both conics the point of concurrence is O.

Now the lines BC, $B'C'$; XO, XX' are harmonically conjugate, so their poles for S' are harmonically conjugate. Since the poles of BC, XO, XX' are A, O', O, it follows that the pole of BC is A', so $AB'CA'BC'$ is a hexagon satisfying the second condition.

3. It is well known that if S and S' satisfy the first condition for $m = 4$ then, by a suitable choice of absolute points, S' and S may be specialised to a conic and its director circle: O is then the common centre and $XX'O'$ is the absolute line. The argument may then be slightly simplified by using the fact that the whole figure is symmetrical about O.

4. I wish to thank Dr. J. A. Todd for his helpful suggestions.

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68. On the quadratic equation

The general solution of the Quadratic Equation, its graphical solution and the relation between the coefficients and roots are the common introduction to the solution of equations of degree higher than the first. We have

$$Q(x) \equiv ax^2 + bx + c = 0$$

leads to

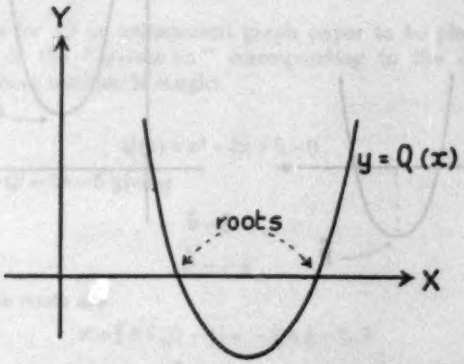
$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

and if these roots are α and β then

$$\alpha + \beta = -b/a$$

$$\alpha\beta = c/a$$

and graphically



If the graph of $y = Q(x)$ touches the X-axis, then there is only one root; and if it neither touches nor cuts this axis there are no roots. Algebraically, the notions of double (or equal) roots, and of complex roots establish that a quadratic equation may be regarded as always having two roots. Graphically, the matter is not usually pressed further.

The general quadratic equation with real coefficients may be re-written, without loss of generality, thus

$$Q(x) \equiv x^2 - 2px + q^2 = 0$$

By the usual method we find the roots to be

$$x = p \pm \sqrt{(p^2 - q^2)}$$

which we will now rewrite

$$x = p \pm i\sqrt{(q^2 - p^2)}$$

Also by the usual method we find that Q has a minimum $\mathring{Q}$ at $\mathring{x}$ where

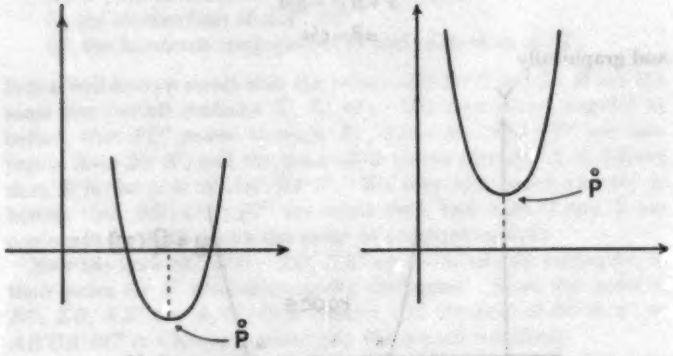
$$\mathring{x} = p$$

$$\mathring{Q} = q^2 - p^2$$

We may, therefore, write the solution of the quadratic as

$$x = \mathring{x} \pm i\sqrt{\mathring{Q}}$$

This may be seen graphically



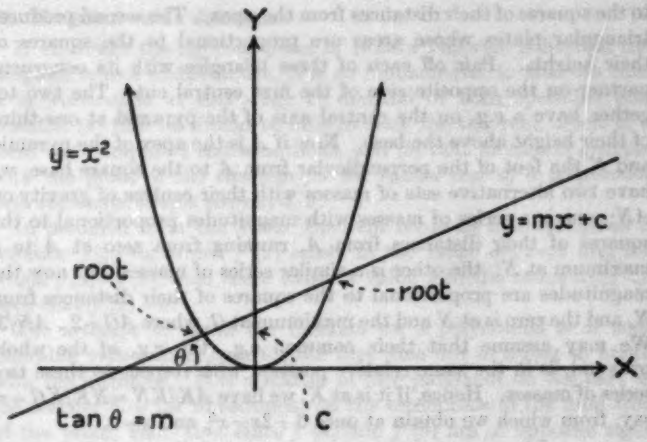
In both cases the " minimum " point $\mathring{P}$ has co-ordinates $(\mathring{x}, \mathring{Q})$; and this graphical solution is valid whether the roots are complex, equal or real.

We observe that p is the arithmetic mean of the roots, and q the geometric mean. The -2 reflects both the alternation of sign associated with the elementary symmetric functions of the roots and the binomial coefficients; the q^2, rather than q, maintains the weight of each term as two.

The general quadratic with real coefficients may also be re-written, without loss of generality, thus

$$Q(x) \equiv x^2 - mx - c = 0$$

Its solution may then be seen graphically as the intersection of $y = x^2$ (the easiest of all parabolas to draw) and $y = mx + c$ (one well known standard form of the straight line). The parabola may be



drawn once for all on transparent graph paper to be placed over the graph of the "$y=mx+c$" corresponding to the quadratic equation whose solution is sought.

Example:

$$Q(x)\equiv x^2-5x+6=0$$

We have $Q'=2x-5$ giving

$$\dot{x}=\tfrac{5}{2}$$

$$\dot{Q}=-\tfrac{1}{4}$$

and thus the roots are

$$x=\tfrac{5}{2}\pm i\sqrt{(-\tfrac{1}{4})}=-\tfrac{5}{2}\pm\tfrac{1}{2}=2,3$$

The solution is given graphically by the cutting of $y=x^2$ by $y=5x-6$; the coordinates of the point of cutting are (2, 4) and (3, 9)—the X-coordinates being the 2, 3 (the roots of the quadratic) and the Y-coordinates their respective squares.

Stoke Rochford. ROBERT ILLING

69. To find the c.g. of a pyramid without integration

A square pyramid can be cut into slices in two distinct ways—by planes parallel to the base, or by planes parallel to the plane through the apex and a pair of opposite corners of the base. The first method of slicing produces thin square plates whose areas are proportional to the squares of their linear dimensions, and therefore

to the squares of their distances from the apex. The second produces triangular plates whose areas are proportional to the squares of their heights. Pair off each of these triangles with its congruent partner on the opposite side of the first central cut. The two together have a c.g. on the central axis of the pyramid at one-third of their height above the base. Now if A is the apex of the pyramid and N the foot of the perpendicular from A to the square base, we have two alternative sets of masses with their centres of gravity on AN; one is a series of masses with magnitudes proportional to the squares of their distances from A, running from zero at A to a maximum at N; the other is a similar series of masses, but now the magnitudes are proportional to the squares of their distances from N, and the zero is at N and the maximum at G, where $AG = 2 \cdot AN/3$. We may assume that their common c.g., the c.g. of the whole pyramid, is in the same relative position with respect to these two series of masses. Hence, if it is at K, we have $AK/KN = NK/KG = x$, say, from which we obtain at once $3 + 2x = x^2$, and $x = 3$.

C. DUDLEY LANGFORD

70. Newton's theorem

Newton's theorem states that if $x_1, x_2, \ldots, x_n$ are the roots of

$$x^n + p_1 \cdot x^{n-1} + \ldots + p_{n-1}x + p_n = 0$$

and $S_r = \sum x_1^r$, then if $r \leqslant n$

$$S_r + p_1 S_{r-1} + \ldots + p_{r-1} \cdot S_1 + r p_r = 0$$

The proofs of this theorem given in most algebraic texts are based on the principle of expressing $f'(x)$ both in terms of the roots $x_1, x_2, \ldots x_n$ and also the coefficients $p_1, p_2 \ldots p_n$ and then equating the coefficients. Since there appears to be no essentially different way of proof given in these textbooks the following alternative proof may be of some interest.

Consider the following equalities

$$S_r = (-p_1)(S_{r-1}) - \sum x_1 \cdot x_2^{r-1}$$

$$\sum x_1 \cdot x_2^{r-1} = (p_2)(S_{r-2}) - \sum x_1 \cdot x_3 \cdot x_3^{r-2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum x_1 \cdot x_2 \ldots x_{r-1}^2 = \{(-1)^{r-2} \cdot p_{r-2}\} \cdot \{S_2\} - \sum x_1 \cdot x_2 \ldots x_{r-1}^2$$

$$\sum x_1 \cdot x_2 \ldots x_{r-1}^2 = \{(-1)^{r-1} \cdot p_{r-1}\} S_1 - r \cdot (-1)^r \cdot p_r$$

Multiply the first equality by $+1$, the second by -1, the third by $+1$ etc. and add. On cancellation of terms common to both sides. Newton's relationship, (i.e. $S_r + p_1 S_{r-1} + \ldots p_{r-1} \cdot S_1 + r p_r = 0$) is obtained.

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71. The mean value theorems

1. Although the rigorous proof of Rolle's Theorem is beyond the range of school mathematics, it is so fundamental to analysis that it is worthwhile at this stage to demand of both specialist and scientist (*a*) that he should be able to state the theorem, albeit in a form with simplified end-conditions, and (*b*) that he should grapple with the proof of the theorem *for the special case of a polynomial with successive simple zeros at a, b.*

In modified form the Rolle Theorem reads as follows: given a function $f(x)$ for which $f'(x)$ exists in the range $a \leqslant x \leqslant b$ and for which $f(a) = f(b)$, then there exists a value ξ in the range $a < \xi < b$ such that $f'(\xi) = 0$.

The attempt to prove the theorem for a polynomial is of considerable value, since here for the first time one is trying to break with the intuitive. It is a matter of nice judgment on the part of the teacher how far to comment on the credentials of the final step of the proof, viz.: that since $f'(a)$ and $f'(b)$ are of opposite signs this function vanishes at some intermediate point.

2. When Rolle's Theorem is understood, the proofs of the mean value theorem depend in each case on finding a " Rolle function ".

(*a*) The first theorem, viz.: where $f'(x)$ exists in $a \leqslant x \leqslant b$ then for some ξ in $a < \xi < b$ we have $f'(\xi) = [f(b) - f(a)]/(b - a)$.

Clearly we can construct a function $y = mx + c$ which has values $f(a), f(b)$ at $x = a, b$. Then $f(x) - mx - c$ is a Rolle function and the result follows.

(*b*) For Cauchy's Theorem it is not difficult to elicit from a class a Rolle function by cross-multiplication as

$$\phi(x) \equiv \{f(b) - f(a)\}g(x) - \{g(b) - g(a)\}f(x).$$

Then $\phi(a) = \phi(b)$ and the result follows.

3. The proof of the second mean-value theorem seems to the writer to be one which is to be seen rather than remembered. (This opinion may be the result of failing to remember it in a Cambridge Scholarship examination.)

The following proof may be of interest: it is stated in Maclaurin form. *To prove* that if $f'(x), f''(x)$ exist in the range $0 \leqslant x \leqslant h$ then $f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(\xi)$ where $0 < \xi < h$.

Before commencing it needs to be emphasised for the beginner that in Rolle's and derived theorems there is shown to be at least one value ξ *inside* the range. A diagram shows that the vanishing of $f'(a)$ and/or $f'(b)$ does not affect this result.

Proof. Our knowledge of expansions would lead us to expect that for some value of K we should have

$$f(h) \simeq f(x) + (h - x)f'(x) + (h - x)^2 K$$

We shall write, exactly, as a definition of $\phi(x)$ the identity in x:

$$f(h) \equiv f(x) + (h-x)f'(x) + (h-x)^2 K + \phi(x) \qquad (1)$$

Clearly $\phi(h) = 0$; and we can choose K so that $\phi(0) = 0$. Then $\phi(x)$ is a Rolle function for the interval, giving $\phi'(\xi) = 0$, where $0 < \xi < h$.

But by differentiating (1) we have

$$0 \equiv (h-x)f''(x) - 2(h-x)K + \phi'(x)$$

Substituting ξ for x, using $\phi'(\xi) = 0$ and $\xi \neq h$ we have

$$f''(\xi) = 2K$$

Substituting in (1) and writing $x = 0$ we have, since $\phi(0) = 0$, the result

$$f(h) = f(0) + hf'(0) + \tfrac{1}{2}h^2 f''(\xi)$$

A. J. MOAKES

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,—In your issue of October 1959 Misses Curphey, Kelley and Moffat (CKM) have given a table of the frequency distribution of the digits in the first 10,000 decimal places of π. Whereas their table was intended to replace the inaccurate one of Messrs. Doe, Ogden and Vieri that appeared in the February 1959 issue of the Gazette, this itself is defective because of the inaccuracy, after the 7480th decimal place, of the value of π which seems to have been employed for this enumeration. Same is the case with the observations made by Mr. A. Kodym on some of the numerical properties of these digits and reported in the February 1960 issue.

I have recently investigated in some detail the statistical aspects of the 10,000 digits of π appearing in "Chiffres" 1, 17–22 (1958). The complete agreement of this value with the correct one, as re-computed by Mr. Felton, has been attested by Prof. J. C. Miller (Math. Rev. 20, Abs. No. 1436, 1959). The frequency distribution of the digits in $\pi - 3$ is seen to be as follows:

Digits / Decimal places	0	1	2	3	4	5	6	7	8	9
1–7,000 (CKM)	657	733	692	686	702	730	708	694	680	718
7,001–8,000	97	100	119	95	107	104	108	92	84	94
8,001–9,000	101	103	100	103	101	99	98	97	90	108
9,001–10,000	113	90	110	90	102	113	107	87	94	94
1–10,000	968	1026	1021	974	1012	1046	1021	970	948	1014

Next, with a view to study the degree of serial correlation among the neighbouring digits, a frequency count of the 10,000 (overlapping) pairs ij of the digits has been done. The observed frequencies f_{ij} of the different types of pairs are given in the following table.

i \ j	0	1	2	3	4	5	6	7	8	9	Total
0	85	103	98	103	98	89	101	93	83	115	968
1	99	99	103	102	121	95	106	90	98	113	1026
2	101	115	110	99	82	118	100	101	100	95	1021
3	102	92	86	94	114	100	90	102	97	98	975
4	95	100	100	89	102	110	103	108	101	104	1012
5	92	117	110	96	108	96	115	107	96	109	1046
6	107	95	117	97	101	124	91	101	90	98	1021
7	89	105	99	91	92	101	95	97	103	98	970
8	86	97	99	93	96	106	114	83	80	93	947
9	112	103	99	110	98	107	106	88	100	91	1014
Total	968	1026	1021	974	1012	1046	1021	970	948	1014	10000

As it should be, $\Sigma_i f_{im} = \Sigma_n f_{mn}$ for all m except when it is either 3 or 8; the difference ± 1 in these two cases is due to the "end" effects of the open array. The smallest frequency in this table is 80, which is for the pair (88), and the largest one is 124, for the pair (65). Further, we note that out of the one hundred entries as many as seventy show a deviation of less than 10 in magnitude from the expected value of 100, seven deviate exactly by that amount and the remaining twenty-three depart by greater amounts.

Detailed statistics of these digits, as obtained by subjecting them to a larger number of standard tests of randomness, is under study and will be reported elsewhere.

Yours Faithfully, R. K. PATHRIA

FOR SALE

Mathematical Gazette, Nos. 337 to 349; Collection of mathematics text-books by Routh, Edwards, Hall and Knight (with key), Hobson, Durell, Forder, etc. Replies to L. Malone, 17 West Oakhill Park, Liverpool 13.

RECREATIONAL MATHEMATICS MAGAZINE. Edited by T. S. Madachy. Subscription rates: $3.00 p.a. for teachers, students and libraries; $3.50 p.a. for the general public. All correspondence should be sent to The Editor, Recreational Mathematics Magazine. Box 1876, Idaho Falls, Idaho, U.S.A.

To the Editor of the *Mathematical Gazette*

DEAR SIR,—May I clarify some matters arising from Prof. John Satterly's letter (Math. Gaz. 44 (350) 1960, December page 296)? I open with a quotation from a Bill now before Parliament:

"9 Eliz. 2 WEIGHTS AND MEASURES

A Bill intituled an Act to make amended provision with respect to weights and measures, and for commercial purposes. A.D. 1960"
Part I, Clause 1, (1):

"The yard or the metre shall be the unit of measurement of length and the pound or the kilogram shall be the unit of measurement of mass by reference to which any measurement involving a measurement of length or mass shall be made in the United Kingdom: and

(*a*) the yard shall be 0·9144 metre exactly;

(*b*) the pound shall be 0·453 592 37 kilogram exactly."

It is made clear elsewhere in the Bill that metre and kilogram are exactly equal to the internationally defined *mètre* and *kilogramme* respectively and that pound and kilogram are lawfully written lb and kg. The same definitions came into use throughout the Commonwealth for purposes of science and technology on 1st July 1959 and the Federal Register of that date made them effective in the U.S.A. for virtually all purposes. There can be no doubt therefore that the unified pound and kilogram now to be used are, like the *kilogramme*, units of mass. The "weights and measures" in the title of the Bill must be presumed to relate merely to the material objects used to ascertain masses and lengths.

In the case of units of force there is no Statutory document settling the matter. It has, however, been well covered by the British Standards (BS 350: Part 1: 1959 and BS 1991: Part 1: 1954 when read with Amendment No. 3 of 4th October 1960) and by the International Organisation for Standardisation (covering Anglo-American and metric units and having physicists, mathematicians, engineers, chemists and metrologists from 20 countries at the plenary sessions of the relevant committee) as well as by the *Conférence Générale des Poids et Mesures*, the supreme arbiter for metric units. The two bodies first mentioned agree that in addition to the units of force called dyne (dyn), newton (N) and poundal (pdl)—which correspond to g cm s^{-2}, kg m s^{-2} and lb ft s^{-2} respectively—there exist "technical" units of force called kilogram-force (kgf), pound-force (lbf), etc. which are the forces on bodies of mass 1 kg, 1 lb, etc. when these have the standard acceleration, defined exactly to be $g_n = 980{\cdot}665$ cm s^{-2}. The BSI defines analogously the ozf and the tonf. Indeed we may say that if XXXXX be the name and X the unit-symbol of any unit of mass then the unit of force called XXXXX-force, and assigned the unit-symbol Xf, is such that 1 Xf/1 X corresponds to g_n, exactly. There also exists a unit of mass called the slug defined as (980·665/30·48) lb; 1 lbf thus corresponds to slug ft s^{-2}.

It will be noted that the kilogram-force, pound-force etc. are defined by means of dynamical principles. If, for introductory teaching, a simpler definition is needed it must be that these forces are the attrac-

tions on bodies at mass 1 kg, 1 lb, etc. in a standard gravitational environment which, may, if required, be defined as " a place where a freely falling body descends 490·3325 cm in the first second from rest ". So-called gravitational " units "—such as the lbwt defined as the *local* attraction on a body of mass 1 lb are eschewed since they depend on the *local* initial acceleration in free fall, g, and are thus not constant quantities. It may soon become significant that if the tensile strength of a steel is 30 tonwt/in² at London it will be 40 tonwt/in² on Mars! It will, of course, be about 30·02 tonf/in² everywhere.

Reference is made in Prof. John Satterly's letter to the different customs of engineers and physicists. Most technologists and scientists now use symbols for quantities (which are regarded a product of numerical measure and unit quantity) rather than for the mere measures of these quantities. If a body have mass M and velocity u then its kinetic energy is $Mu^2/2$ and this is the most convenient formula for engineers and physicists alike. Anyone who cares may write this

$$Wu^2/2g \quad \text{or} \quad W_n u^2/2g_n, \quad (\text{where } W = mg \quad \text{and} \quad W_n = mg_n)$$

but this sort of thing is dying out. Whichever of the 3 formulae mentioned be used, it is not dependent on the units to be used, since the symbols stand for quantities which will carry their units with them thereby bringing in any conversion factors required.

British and American engineers (except for a few aeronautical engineers) have always regarded the pound (lb) as a unit of mass but—most deplorably—they generally used also to write the same name and unit-symbol for the unit of force now called pound-force (lbf). By adopting distinct names and unit-symbols for the units of force they have removed ambiguity. (A few authors have also written kgm and lbm instead of kg and lb for units of mass—a deplorable practice contrary to national and international law and standards and conflicting with a convenient device mentioned later.)

It is greatly to be desired that we should abandon non-metric units and that engineers of all lands will abandon the technical units and use only the newton and dyne and their decimal multiples and submultiples. In the meantime the Standards provide units understood by technologists while at the same time avoiding ambiguity and inconsistency.

Those who find it unfortunate that the name of a quantity, " force " is now built-in to the names of the units kilogram-force, pound-force etc. may care to adopt a device advocated by the present writer. This is to write, instead, kilogram fors (kg f), pound fors (lb f), etc. and to regard the fors (with unit-symbol f) as a unit of specific force (i.e. force per mass) defined as exactly 980·665 dyn/g and therefore corresponding to the standard acceleration g_n. This device has, at present, no official or formal backing. It is evident, however, that it does not conflict with law, and Standards.

The use of kgf and lbf is growing both in industry and in university engineering. They now appear in the Examination papers for B.Sc(Eng.) of the University of London. Unfortunately the schools and the school-examining bodies lag behind. If the students reached us, the University

teachers, having heard only of the dyne, the newton and the poundal we could well impart information about the other units of force. Better still they should already have appreciated the use of the kgf and lbf. Unfortunately however we generally have to start by trying to erase the belief that engineers are still misguided enough to use pound (lb) both for mass and force or to use a variable quantity (lbwt) as a unit. I hope Sir, that through your valued publication I may thus not only add to the historical matter provided by Professor Satterly but also encourage applied mathematicians and physicists in schools to teach in a manner not discordant with the reforms that have been made.

Yours faithfully, E. J. Le Fevre

Queen Mary College,
London, E.1.

REVIEWS

Building up Mathematics. By Z. P. Dienes. Pp. 124. 16s. 1960. (Hutchinson).

Dr. Dienes is a reformer impelled by iconoclastic zeal. In an area so wide as that of mathematical teaching in this country it is obviously possible to find many targets for criticism, but Dr. Dienes goes further. He suggests, or at least the reviewer understands him to suggest, that the whole of traditional mathematical education is faulty. Thoroughgoing rethinking and reconstruction of the methods and aims of the curriculum are called for.

The author begins by criticizing existing aims. These, he says, are as follows. (*a*) To give skill in computation needed in everyday life. But apparently very little of this is really needed; all the rest will be done by automatic computers. (*b*) To train the mind to think logically. "But what could be more illogical than to perform a great number of quite unintelligible acrobatics with symbols for the simple reason that you would get a detention if you did not?" (*c*) To provide basic techniques for scientific development. But—and here the argument does not seem very clear—to do this we must teach understanding of processes, to make them generally applicable. Only a few pupils have the capacity to understand fully enough to make them expert in the application of mathematics, so that this aim leads to greater specialisation: teaching more mathematics to fewer people. "The mathematically fit survive by natural selection; the rest get gradually relegated to the mathematical lumber-room as second-class citizens unfit for initiation into the mysteries." (*d*) To build up personality. But for children who do not enjoy mathematics the punishment-reward system is disintegrating, while for those who do, operating within closed sets of techniques can have little integrating effect.

By this time most practising teachers will be thinking that they would like to introduce Dr. Dienes to some of their pupils in their lower streams. What remedy does the author propose? Basing his theory on the work of Piaget, Bruner, and Bartlett, Dr. Dienes outlines the process of learning mathematics as a cyclic scheme in which concepts are learnt and then used as the basis for further concepts. Each concept must first be experienced in preliminary play, then recognized as the common element in a wide variety of structured games, then used and practised until in turn it can be made the basis of games leading to recognition of a further concept. "It will be clear", says the author, "that the kind of mathematics-learning described here is not one that very often takes place in the conventional class-lesson." Instead he recommends work (or play) in groups of two or three, using assignments and much concrete material. Detailed consideration is given in the chapters that follow to the materials used for formation of various concepts: place-value in arithmetic (Multibase Arithmetic Blocks); distributive laws in algebra (balance with hooks and rings); equations (pegs, strips and squares); factorisation (squares and triangles, strips and pegs); exponentiation and logarithms (Blocks again); functions (charts and graphs).

It is quite clear that this is all going to be great fun for somebody; it is equally clear that it will need an exceptionally skilled teacher to ensure that very much is learnt from it. The thought of what could happen in a classroom littered with blocks of all shapes and sizes removed from their " radix " boxes will make many teachers hesitate. Most will suspect that the huge private joke which has evidently come off in illustration V has nothing whatever to do with factorizing $A^2 - 5A + 6$. But it would be wrong to dismiss the book as visionary. Too many of us are conscious that, in spite of our best efforts, mathematics to many of our pupils is a game played with meaningless marks on paper. This book may help us to find out some of the reasons why. But, in the reviewer's opinion at least, a wholehearted reformation might lead to the equal danger of mathematics becoming a meaningless game played with blocks, balances, pegs and rings, if carried out exclusively on the lines advocated in this book.

H. MARTYN CUNDY

National Certificate Mathematics Vol. I. By P. ABBOTT and C. E. KERRIDGE. Revised by W. E. FISHER. Pp. xii, 410. 9s. 6d. 1960. (English Universities Press)

This is a " completely revised " edition of a book which was first published in 1938 and which has been deservedly popular as a textbook for part-time students in Technical Colleges. The revision has been made because " advances in technology have been reflected in changes in courses ". At this level, of course, the impact of these changes has not been very great, and it is not surprising to find that little alteration has been made to the textual content. Some extra worked examples are included, mainly chosen from mechanical engineering. Considerable extensions have been made to the exercises at the ends of the chapters, problems being selected from a wide variety of examination sources. The reviser has done a useful job, with limited scope. It will be interesting to see whether more extensive improvements can be made to the succeeding volumes.

J. F. REED

Engineering Mathematics. By J. BLAKEY and M. HUTTON. Pp. 603. 40s. 1960. (Blackie and Son Ltd.)

The mathematical training of to-day for the engineers of tomorrow is not a subject on which syllabus compilers find decisions easy and this text will be scanned by many with more than usual interest.

From the standpoint of content *Engineering Mathematics* divides neatly into two portions stemming from separate origins. The first has resulted from a condensation of Dr. Blakey's well-known *University Mathematics* from which, by omitting chapters on the polar and general equations of a conic and on solid geometry, the authors have produced a useful text comprising much of the mathematics

needed by engineers; it is treated by traditional methods with traditional standards of rigour. Centres of gravity, moments of inertia, spherical trigonometry and harmonic motion receive brief, but adequate treatment; one notes that the two chapters on differential equations devote some dozen pages to *D*-operator methods, that determinants are included but, as indicated above, 3-dimensional coordinate geometry is out.

The remaining section of the book consists of four final substantial chapters containing the more forward-looking parts of the syllabus. In these chapters, written by Dr. Hutton, the matter has been selected from numerical analysis, statistics and operational calculus, with an eye to the specific needs of engineers. Certainly every engineer needs a modicum of statistical knowledge and the need for numerical calculus in this age of complex problems and powerful computers is developing rapidly—in fact few will wish to quarrel with any of the inclusions. This having been said it should be noted that many who strongly approve the teaching of numerical methods will consider the omission of matrices and of any basic finite differences rather as Hamlet without the Prince of Denmark. No doubt the authors weighed carefully the factors of time and syllabus content but the results of the omission are serious. Thus for linear simultaneous equations the engineer is left with the not very efficient rule of Cramer or with the use of relaxation, and where finite differences might help—as with numerical quadrature, most important to the engineer—the subject is just excluded from the text. Even at the cost of pruning earlier chapters at least one of these powerful symbolic tools might have been introduced, and preferably finite differences in this context.

The first chapter of this final section deals with numerical solution of equations (polynomial and transcendental): it contains an interesting derivation of a convergence criterion for an iterative process. An example of inverse interpolation is necessarily limited to the relatively weak " rule of false position ".

The remainder of the numerical work is contained in a useful chapter on relaxation methods which proceeds as far as the solution of boundary-value problems of Laplace and Poisson type having rectangular boundaries. A solution of the latter type depicting eight stages of the calculation, with explanatory text and diagrams, should prove invaluable to the student wishing to test his skill at this art.

Can sufficient statistics be provided in thirty pages? The reviewer believes that many students will be grateful to Dr. Hutton for this easy introduction and for the manner in which by skilful use of examples and intuition he leads from the simple elements of probability to binomial, normal and Poisson distributions. Finally electrical engineers and others have not been forgotten and the last chapter of the book is devoted to operational calculus. Here the Laplace transform is applied to the solution of differential Equations of engineering importance. Heaviside's unit function and Dirac's impulse function receive brief treatment, once more chief emphasis being given to their practical application.

Two minor criticisms might be made. The index which well serves

the traditional section becomes very thin in references to the newer parts of the book. Further, for this is a book for universities, there are no references for further reading—a defect which might be especially felt by a student reading the later chapters.

Viewed as a whole this book is well-produced and reasonably priced: above all it is a courageous well-planned presentation of the basic mathematics required by the engineering student of to-day.

J. KERSHAW

Additive Primzahltheorie. By LOO-KENG HUA. DM. 29.50. 1959. (Teubner, Leipzig)

This book was originally written in Chinese, translated into Russian, and the present edition is a translation from Russian into German, revised by the author. The book deals with the Goldbach-Vinogradov theorem that every large odd integer is a sum of three primes and the extension to the Waring-Goldbach problem

$$n = p_1^k + \ldots + p_s^k,$$

i.e. the representation of integers as sums of powers of primes. It also contains a number of theorems, mostly due to Vinogradov, on exponential sums extended over primes.

Written in the traditional Landau style, the book is clear and easy to read, but likely to appeal to the specialist only. An extensive bibliography would have been a valuable addition.

H. HEILBRONN

Analytic Function Theory I. By E. HILLE. Pp. xi, 308. $6.50. 1959. (Ginn)

A new text on complex variable theory must jostle with many existing competitors. In Hille's second volume, he may be able to show us more novelties, but here the main content must be more or less fixed, and the author must show his skill through form and arrangement.

Hille believes that " the Cauchy integral is a much more pliable and versatile tool than the power series when it comes to doing things in function theory ", and hence this volume is built round complex integration. But there is no undue haste to get to Cauchy's theorem; the way is prepared by a careful study of the elementary functions and their geometrical properties, and here the power series plays an important part. In the introductory chapter on number systems, and in places throughout the volume, the influence of the modern axiomatic formulation of mathematical theories is evident; so also is the author's wish to exhibit function theory as a part and an illustration of the profounder doctrine of functional analysis. There is an excellent chapter on representation theorems, in which the Taylor and Laurent expansions serve to analyse the structure of meromorphic functions and integral functions, and for illustration the results are applied to the

Gamma function. The chapter on residues exhibits the power of this calculus not only in the evaluation of real integrals but also in expansion and summation problems, and in the study of inversion problems, such as the structure of w as a function of z defined by a relation $F(z, w) = 0$. This section clearly owes much to Lindelöf, whose *Calcul des Résidus*, now over 50 years old, is, says Hille with justice, " still the best in the field ".

Some very praiseworthy smaller points are: the fresh and interesting exercises for the reader; the historical footnotes which pay tribute to the founders and masters of this fascinating topic; the clear and flexible English style, happily free from pretentious barbarisms.

Here is an admirable start to Ginn's new series of " Introductions to higher mathematics "; Hille's second volume will be eagerly awaited by all who need to learn or to teach this subject.

T. A. A. BROADBENT

Elementary Statistics with Applications in Medicine and the Biological Sciences. By FREDERICK E. CROXTON. Pp. vii, 376. $1.95. 1959 (Dover Publications, Inc., New York.)

This book is well described by its title. It deals with the graphical presentation of data, calculation of moments, correlation, regression, and the standard simple estimation procedures and significance tests (t, F, χ^2). The whole is done very carefully and lucidly; the explanations are good, and the warnings given from time to time are helpful. The mathematical standard assumed for the reader amounts only to a knowledge of simple algebra, such as an average biologist might possess.

There are however a few minor blemishes. It is not very clear from the text what exact purpose is served by calculating a regression line: certainly the reader may be a bit surprised by the regression on p. 118, which shows that a patient injected with no glucose will retain ·046 units in his body! It is difficult to understand the assertion on p. 239 that confidence limits for a difference are meaningful only when the difference is significant—surely the philosophy of confidence limits is that if they are calculated in *every* relevant case, they will contain the true value in a stated proportion of cases. The estimation and significance procedures for variances described on pp. 287–295 can be very misleading if the population has a non-normal distribution. And the statement on p. 309 that " the estimated variance between column means cannot be considered significant unless it is large in relation to the estimated interaction " is questionable. The data in question show an interaction between dressings (columns) and times (rows) in the healing of wounds. Since if dressings had no effect at all, they could not interact with times, the existence of the interaction logically demands the existence of an effect of dressings (even if one did not know from common experience that the choice of dressing influences the rate of healing of a wound). In fact, the important issue seems to be not the " significance " of the difference between dressings, but the choice of the best one: and that is rather a different question.

CEDRIC A. B. SMITH

Exposé Moderne des Mathématiques Elementaires. By LUCIENNE FELIX. Pp. 421. 2,900 F. 1959. (Dunod, Paris)

Fundamental structures, groups, rings, fields and vector spaces are illustrated in the familiar settings of arithmetic, algebra and geometry, each constructed on an axiomatic basis. A serious attempt is made to find an intuitively satisfying axiomatisation of metrical geometry with symmetry as a fundamental concept. The chapter on analysis is not up to the high standard of clarity of the rest of the book; the definition of $\lim_{x \to x_0} f(x)$ omits the necessary inequality $0 < |x - x_0|$ and the proof of the chain rule for differentiation is defective.

R. L. GOODSTEIN

Galoissche Theorie. By E. ARTIN. Pp. 86. DM. 5.30. 1959. (B. G. Teubner, Leipzig)

This is a translation by V. Ziegler of Artin's *Notre Dame Mathematical Lectures* volume on Galois Theory with improvements by the author. The major changes are in the proof of the Fundamental Theorem and a proof of the irreducibility of the cyclotomic polynomials. The third of the three parts, (by A. N. Milgram) is on applications of Galois Theory (solution of equations by radicals, straight line and circle construction).

R. L. GOODSTEIN

Einführung in die Mathematische Logik. Part I. By G. ASSER. Pp. 184. DM. 11.25. 1959. (B. G. Teubner, Leipzig)

This is the first of a projected three volume work by a former pupil of Schröter in Berlin. The present volume is entirely concerned with two-valued sentence logic. The treatment is very thorough and comprehensive. Amongst the topics considered are truth table valuations; normal forms; axiomatisation and completeness, consistency and independence of axioms; natural inference. The second and third parts will deal with predicate logic and set theory.

R. L. GOODSTEIN

Fonctions Spheriques de Legendre et Fonctions Spheroidales, Tome 1. By L. ROBIN. Pp. xxxv + 201. 4000 Fr. 1957. (Gauthier-Villars).

This is the first volume of a three-volume work, the plan of which is outlined by the author in a twenty-seven page introduction to this volume. The present volume treats Legendre polynomials and functions of the second kind of positive integral degree, associated functions of integral order and degree, and spherical harmonics of integral degree. It is proposed that Vol. II will discuss associated functions of general type, asymptotic formulae, and orthogonal expansions involving Legendre functions, and that Vol. III will deal with addition formulae, theorems about zeros of the functions, applications to the equations of

mathematical physics, and the properties of related functions such as Gegenbauer functions and spheroidal functions.

Although the book is written from the point of view of the applications of the Legendre functions, the author does not in any sense sacrifice rigour. He has written a full and clear account of the properties of the elementary Legendre functions in this volume. The result is a text which can be read easily by anyone wishing to acquire a working knowledge of these functions. For anyone who is already acquainted with the theory it will serve as a useful work of reference.

If the remaining two volumes of the set are of the same high quality as this first one, the completed work will be the most important to appear since the publication of Hobson's treatise in 1931 and may well supersede it as the standard work on the subject.

I. N. SNEDDON

Differential Equations. By F. R. MOULTON. Pp. 395. $2.00. 1959. (Dover Publications)

Moulton's book, which first appeared in 1930, is well known to serious students of the theory of differential equations. Devoted to the study of certain aspects of the theory of ordinary equations, it combines rigorous treatment with the study of difficult practical problems mostly of a physical nature. It has now been re-issued by the Dover Co. in a paperback edition. It is unlikely to be used as a textbook for students but it is invaluable to lecturers preparing notes on the subject.

I. N. SNEDDON

Mathematics in Physics and Engineering. By J. IRVING and N. MULLINEUX. Pp. 883. $11.50. 1959. (Academic Press)

Almost every university teacher of mathematics is involved nowadays, at some level or another, with the teaching of mathematics to students of engineering and physics. The publication of a new book on the subject is bound therefore to arouse interest. However, the volume under review is disappointing and it is difficult to imagine that any experienced teacher of the subject will find much in it to interest him.

The choice of topics made by the authors is most conventional: differential equations, Bessel and Legendre functions, integral transforms, matrices, calculus of variations, complex variable theory and the calculus of residues, numerical methods, integral equations. It will be seen from this list that the chapters cover ground which has been treated excellently in several recent books. There may be some virtue in having all the topics contained within the end-boards of a single volume, but only if the authors' treatment is distinguished by a novel approach. Unfortunately this is not the case in the present volume. Indeed the book reads as if the authors had sent to the publishers a typescript of the lecture notes they had accumulated over the years—an impression which is borne out by the works chosen to be listed in their bibliographies.

The authors have adopted a slightly out-of-date view of the kind of mathematics needed for physics and engineering. The emphasis throughout is on " old-fashioned " analysis. Even their algebra (of matrices) is directed entirely towards numerical calculations. No mention is made of groups and other algebraic concepts necessary to the understanding of many parts of physics. It is strange, too, to find that in a book on the mathematics of physics, the terms " Hilbert space " and " linear operator " are not defined. That the elements of functional analysis can be made intelligible to physicists and engineers has been demonstrated by the books of Bernard Friedman, Sobolev and Vulikh and the authors might well have followed their example. Instead, even in the chapter on Legendre functions, the authors do not take the opportunity of introducing the idea of an orthonormal system of functions.

Within their own limits the authors do not produce a satisfactory book. The treatment of differential equations, both ordinary and partial, is very sketchy indeed—for instance, there is no attempt to even describe the nature of the arbitrary elements entering into the solution of differential equations. (In fact, on p. 69, the definition of a linear ordinary equation is defective). Non-linear ordinary differential equations—a subject of great interest to engineers—are treated quite inadequately. The development of the theory of the calculus of residues is also faulty and misleading in that it does not state the difficulties which are being glossed over. Many important topics are relegated to appendices and treated all too briefly there. The account of tensors will be of very little help to the physicist trying to read the theory of relativity or the engineer studying the mechanics of continua. Cauchy's theorem is also put in an appendix, the proof depending on the divergence theorem which itself gets a remark that it " follows almost immediately from the definition of div **F**" by way of proof!

The result is a thoroughly conventional book which will do the average student no harm but which contains material which is much better presented in many shorter books, and, if his tastes run to cyclopedic surveys, in the standard texts by Margenau and Murphy, Jeffreys and Jeffreys, and Morse and Feshbach. One bad feature of the book which might do harm to an immature reader is the bibliography. At the end of each chapter there are listed general references. These are very much out of date. For instance, the latest book on partial differential equations mentioned is that of Webster who died in 1926. Even if restricted to works available in English the author of a book with a title such as the present one must surely encourage a student to read the standard books by Courant and Hilbert, Sommerfeld and the authors mentioned above.

The book is certainly well printed and bound and, considering that it contains 850 pages, is not expensive, but cannot be recommended to the class of reader for whom it is intended.

I. N. SNEDDON

Operational Mathematics. 2nd. edition. By R. V. CHURCHILL. Pp. 337. 54s. 6d. 1958. (McGraw-Hill, London)

This book is a revision of the author's " Modern Operational Mathematics in Engineering " which is well known to teachers of engineering mathematics. The same general framework is preserved but new material has been added to each of the ten chapters. In the early part of the book new proofs of some of the results have been given and fresh examples added. It is in the last chapter that the major changes have taken place. Material about " generalized " Fourier transforms has been added and the whole thing kept up to date.

With the coming of the new edition the publishers have introduced a much more attractive format. The result is the ideal book for students who wish to learn the elements of the Laplace transform and the operational calculus.

I. N. SNEDDON

Alfred Haar, Gesammelte Arbeiten. Edited by B. SZ.-NÁGY. Pp. 660. 1959. (Akademie der Wissenschaften, Budapest)

With the edition of the collected work of Alfred Haar (1885–1933) the Hungarian Academy of Sciences has paid a fitting tribute to the memory of this outstanding mathematician. All his 35 publications are photostatically reproduced and according to subject matter divided into 8 sections each of which is introduced by a short appreciation and valuable references to later developments. The most important sections are those on orthogonal functions (the Haar system), on calculus of variation (Haar's lemma), and, of course, the section on continuous groups (Haar measure and integral) which is Haar's outstanding contribution to modern Mathematics.

Most papers are in German language and for those in Hungarian a German translation has been added. The volume is very handsomely produced.

W. W. ROGOSINSKI

Classical Dynamics. By R. H. ATKIN. Pp. 273. 1960. 30s. (Heinemann, London)

This is a textbook written to meet the needs of undergraduate students of Mathematics and Physics who do not want to read Dynamics as a special subject. The range and contents of the book will be apparent from the chapter headings: Historical Foundations, Theory of Vectors, Kinematics of a Particle, Particle Kinetics (4 chapters), Systems of many Particles, Motion of a Rigid Body in a Plane, Impulses, Lagrange's Equations, Inertia Tensor, Rigid Body in 3 Dimensions, Gyroscopic and Non-holonomic Systems, Small Oscillations.

In the text there are solutions to about 100 examples, a large proportion of them taken from the examination papers of London University. At the end of each chapter there are problems, again of the kind used in degree examinations, numbering some 300 in all. There is no doubt

that the author had constantly in mind the needs of examination candidates—some examples are solved by analytical and by vector methods—and a student who finds example-solving difficult can learn, not only from the model answers, but also from hints in the text.

In expounding the theory of Dynamics, however, the author is not so successful. In particular, the treatment of Line Vectors and the exposition of the Kinematics of a Rigid Body are inadequate. Attention may be drawn to p. 61—"We define the angular momentum *about a fixed point O* as the line vector $\mathbf{r} \times m\mathbf{v}$." P. 40—"The formula $\mathbf{v} = \omega \times \mathbf{r}$ is clearly applicable to all points P of the line OP and so we may call ω the angular velocity of the line OP." P. 166, where the instantaneous centre is used as a fixed point. P. 202—"When T does not contain the time explicitly, $T + V = \text{constant}$."

The book is attractively bound and there are few misprints.

E. V. WHITFIELD

Homotopy Theory. By S.-T. HU. Pp. 347. $11.00. 1959. (Academic Press, New York)

This book is a most welcome addition to the literature of algebraic topology. A difficulty hitherto facing anybody who wished to lecture or to research in homotopy theory has been the need to go to many different sources for the basic material of the subject. Moreover the sources in question have not (with the exception of a brief monograph by the reviewer) been written for the benefit of the inexperienced and have (without exception!) been unsuitable for beginners.

The book under review is comprehensive and satisfying and deserves to become a standard reference work. The author assumes only familiarity with singular homology and proceeds to describe in fair detail the main ideas of algebraic homotopy theory. Chapter 3 deals with fibre spaces, Chapters 4 and 5 with homotopy groups and Chapter 6 with obstruction theory. Chapter 7 deals with cohomotopy groups, but the reviewer would have preferred a chapter on S-theory, which appears to be a much more important topic.

The last four chapters, probably the most useful to practising topologists, treat spectral homology theory. Chapter 8 contains Massey's algebraic theory of spectral sequences derived from exact couples and describes the exact couple associated with a filtered differential group. This is applied in the next chapter to the fibre space situation, the treatment following very closely the lines of Serre's thesis. Serre's C-theory is studied in Chapter 10 and applied in the final chapter to the calculation of the homotopy groups of spheres.

The book is not strictly self-contained in that much of the later material depends on results and notions drawn from exercises to earlier chapters; exercises, moreover, which no reader is expected to do! The author admits that, to tackle the exercises, the reader is usually expected to read the indicated papers; the latter, in fact, usually solve the exercises completely. The expedient of describing germane material as an exercise appears dubious and it is particularly disconcerting that

the exercises range in difficulty from the trivial to so Herculean a labour as that of defining the Steenrod squares and verifying the Thom axioms. Indeed the role of the extra structure in cohomology in homotopy calculations requires more space than it is here given: multiplicative structure appears only among the exercises to Chapter 8 and there is no proof that the multiplicative structure defined on the cubical cochains satisfies the axioms. However, we certainly do not criticize the author for leaving some hard work to the reader, nor for expecting the reader to supplement his reading of the text by the study of certain well-chosen original papers.

There are some errors and obscurities. For example, the reference given on p. 57 to a generalized Hurewicz theorem refers to a non-existent section. Again, the proof on p. 299 that C-equivalence is an equivalence relation appears to be incorrect (and the statement of Proposition 4.1 incomplete). The fuss about Lemma 2.2 on p. 312 is difficult to understand since the assertion is trivial using reduced suspension and the Hurewicz theorem. The author's definition of a complete c.s.s. complex conflicts with current usage—this should be explicit.

Material that the reviewer would have included (e.g. combinatorial homotopy theory, semi-simplicial homotopy theory, Postnikov decompositions) has been omitted. But tastes must vary, and, in the main, the author has shown very good judgment in the selection of material. Certainly he has placed us all in his debt by providing a clear exposition to which we can confidently refer in selling homotopy theory to students and colleagues.

P. J. Hilton

Algebraic Curves. By J. G. Semple and G. T. Kneebone. Pp. 301. 1959. 45s. (Clarendon Press: Oxford University Press)

Most branches of mathematics pass through two stages, the creative stage, when new results are obtained rapidly, and the critical stage, when the logical foundations are carefully examined. In algebraic geometry the great creative stage occurred during the nineteenth century and the beginning of the present century, and the critical stage began only quite recently, with the work of van der Waerden in the thirties.

The present book gives a completely rigorous account of the theory of algebraic curves up to the Riemann-Roch Theorem. The authors assume that the reader has studied projective geometry (the notations of the authors' *Algebraic Projective Geometry* are used, so that this new book follows on naturally from the earlier one). They also assume that the reader either knows a good deal of modern algebra or is prepared to work quite hard to learn it; there is a very helpful appendix which gives all the algebraic results used in the book, with references to where proofs can be found.

The treatment is algebraic throughout. Two examples are worth mention, because they are topics in which the methods of analysis are generally used. Firstly, the derivative of a polynomial is defined al-

gebraically; and secondly, infinite power series are formally defined algebraically, without any considerations of convergence being necessary.

I enjoyed reading the whole book, but particularly the last chapter on "Local Geometry in S_2". The classical geometers made great progress with the use of infinitely near points, but many mathematicians felt that there was something rather dubious about them. It is reassuring to know that they can be put on a rigorous algebraic foundation. This chapter explains clearly the distinction between free points and satellite points on a branch of a curve; it is not easy to find an account of this elsewhere.

It is almost inevitable (though many think it regrettable) that in order to make algebraic geometry rigorous it is necessary to make it so algebraic that there seems to be little geometry left. The authors have tried hard to avoid this by keeping as much geometrical terminology as possible.

The book will undoubtedly become the standard work on curves for post-graduate students of algebraic geometry.

E. J. F. PRIMROSE

Reduktionstheorie des Entscheidungsproblems im Prädikatenkalkul der ersten Stufe. By JANOS SURANYI. Pp. 216. 1959. (Verlag der Ungarischen Akademie der Wissenschaften, Budapest)

The decision problem of the first order predicate calculus i.e. the problem of finding a general method for deciding whether a given formula of the predicate calculus is satisfiable or not has been recognised for nearly fifty years as being a central one in logic and mathemathematics. It had from the start been attacked from two directions (*a*) attempting to find larger and larger special classes of formulae for which a decision method could be given, (*b*) attempting to reduce the general problem to that of a special class of formulae. Presumably there was a hope that these methods would converge, that it would be possible to reduce the general problem to that for a special class for which a decision method could be given. However, it must have been realised early that there was little chance of success since a solution would enable one, for example, to decide at once the question of the consistency and independence of such deep propositions as the axiom of choice and continuum hypothesis for certain systems of axiomatic set-theory. The impossibility of a general solution by any mechanical or effective method was formally proved in 1936 by Alonzo Church and, independently, by A. M. Turing. However this does not mean that the study of the decision problem is now pointless and unprofitable but only that the two methods of attack can never converge. In view of the great mathematical value of a solution of the decision problem it is obviously of interest to continue method (*a*) of attack and provide more and more types of formula for which a decision method is known. An excellent survey of this field was given recently by Ackermann (*Solvable cases of the decision problem*, North-Holland Publishing Company, Amsterdam 1954). The present book is entirely devoted

to method (*b*). Apart from its interest for its own sake, this is important for at least two reasons. First, in view of the result of Church and Turing any result of type (*b*) sets a strict limitation on method (*a*); e.g. the fact (theorem XII) that the general problem can be reduced to that of formulae having a prefix of the form $\forall\exists\forall\exists^3$ and containing (apart from singularly predicate variables) at most four binary predicate variables, tells us at once that there is no point in attempting to find a general solution for this class of formulae. Second, any such simplification of an algorithmically unsolvable problem provides a possible tool to be used for proving the unsolvability of some other decision problem, as, for example, unsolvability results about Turing machines were used by Novikoff to prove the unsolvability of the word problem for groups.

The present book is quite self-contained, and starts with a definition of the predicate calculus and the decision problem. This is a field where detailed proofs often leave the reader completely bewildered until he has reversed the long process of rigorous synthesis and disentangled the intuitive ideas behind the proof. Particular praise is due therefore to Suranyi for his deliberate attempt to tell the reader first what the idea of the proof is. In fact his system of leaving out all the more intricate technical details and collecting them together in an appendix is one that should be more widely adopted. Suranyi has collected together practically all known results on reduction types and has in almost every case been able to simplify the proof or sharpen the result. Particularly elegant is his proof of the Skolem normal form theorem by the elimination of existential quantifiers by the introduction of functional variables and the subsequent elimination of these functional variables. There are also one or two previously unpublished results e.g. theorem III: The class of formula in which neither negation nor the constant truth-value F occur is a reduction type with respect to validity (i.e. there is an effective method of associating with each formula A a formula B of this class such that A is valid if and only if B is valid); theorem X: The class of closed formulae of the form $\forall x_1 x_2 x_3 M_1 \wedge \forall y_1 y_2 \exists y_3 M_2$, where M_1 and M_2 are built up only from sentential connectives, individual variables, arbitrary many one-place predicate variables and at most one two-place predicate variable, is a reduction type (for satisfiability).

The book concludes with an account of the recursive unsolvability of the decision problem, followed by a postscript on open problems and an appendix containing technical details and lemmas omitted from the proofs.

J. C. SHEPHERDSON

Unvollständigkeit und Unentscheidbarkeit. By W. STEGMULLER. Pp. 114. 33s. 6d. 1959. (Springer-Verlag, Vienna)

Starting with an informal account of Gödel's incompleteness theorem and the impossibility of an intrinsic proof of freedom from contradiction of arithmetic, the author proceeds to technical details and

discusses primitive and general recursion, Church's theorem on the undecidability of predicate logic, Kleene's generalisation of Gödel's theorem and recursive enumerability.

R. L. GOODSTEIN

Vorlesungen über Höhere Mathematik. By A. DUSCHEK. Vol. 2, 2nd. Edn. Pp. 401. DM 48. 1958. (Springer, Vienna)

The first edition of this work was reviewed in *Gazette* XXXV, p. 221. The extensive revision for this new edition was completed just before the author's untimely death. The contents range from the differentiation and integration of functions of 2 variables, to linear algebra, tensor analysis and differential geometry, with a valuable collection of examples with solutions.

R. L. G.

Mathematical Programming and Electrical Networks. By J. B. DENNIS. Pp. 186. 36s. 1959. (Chapman and Hall, London)

This is a third of a series of monographs, the purpose of which is to present current research studies to interested workers. The present volume is concerned with the relation between mathematical programming (not to be confused with digital computer programming) and electrical networks. It is a specialist work of interest mainly to those who are already familiar with the subject of linear programming. The author first introduces the *general* programming problem. Briefly, this is to minimise an objective function $\phi(x)$ subject to constraints $g(x) \geqslant 0$, $x \geqslant 0$, where x is a column vector of n components and $g(x)$ is a column vector of m functions $g_i(x)$ of x. Any vector which satisfies the constraints is called a *feasible vector*, and that which produces as low a value of the objective function as any other feasible vector is known as an *optimal* vector. If $\phi(x)$ is a linear combination of the x's and the $g(x)$ functions are also linear, then the general problem becomes a *linear programming* problem. Similarly if $\phi(x)$ is quadratic (the $g(x)$ remaining linear) then we have a *quadratic programming* problem. The rest of the introduction deals with questions such as local and global minima, and the relation to the generalised Lagrange multiplier rule.

Most of the book is devoted to developing a correspondence between the linear programming problem and the properties of electrical networks containing diodes. For example the current distribution in a network of resistors and current sources which produces the least heating is the one which satisfies Kirchhoff's equations: where diodes are present some of these equations become inequalities, and thus give rise to a linear programming problem. A formal proof of this equivalence leads to a physical interpretation of some of the main theoretical ideas of programming theory. The network analogues developed in this way show that one could obtain the solution of a programming problem (e.g., the shortest path problem) by actually setting up the electrical model. As the author points out, however, that this is an impracticable

procedure in most cases, firstly because such problems can already be solved economically on digital computers, and secondly because the networks would involve high precision components not readily available. The author claims only that this approach gives a physical insight into the mathematics of the problems, and serves as a guide to developing techniques for their solution. An account of some of these is given.

At the end of the book are some appendices giving formal proofs of relevant theorems in general programming. This is in no sense however, an introduction to the subject: for that one must refer to standard texts, for example, " An Introduction to Linear Programming " by A. Charnes, *et. al.*, (Chapman and Hall, 1953).

R. A. BROOKER

Handbook of Automation, Computation and Control, Vol. 2. Edited by EUGENE M. GRABBE, SIMON RAMO, DEAN E. WOOLDRIDGE. Pp. 900. 136s. 1958. (Chapman and Hall)

This book is virtually an encyclopaedia of computer technology, individual sections being contributed by specialists in their own fields. The book falls into five main sections: (*a*) computer terminology (which is mainly a glossary of terms), (*b*) programming, (*c*) data processing operations, (*d*) design of digital computers, (*e*) design and application of analogue computers, and (*f*) unusual computer systems.

The section on programming has been prepared by John W. Carr (a former President of the Association for Computing Machinery) and is the largest single contribution. It covers all the traditional programming techniques, and also describes the more recent developments in automatic programming. In this approach the computer prepares its own detailed programme of instructions from a statement of the problem in some procedural language close to that of conventional mathematics. The material is mainly drawn from American sources but includes some details of a Soviet algebraic language computer. Other topics covered include microprogramming (the synthesis of the basic machine instructions as a sequence of micro-operations, each of which perform some very elementary task such as complementing a number), and the construction of test programmes and acceptance tests for a computer. There is an extensive bibliography at the end of this section, but again mainly of American and Russian references. The reviewer could find no reference at all to English work in the programming field subsequent to Dr. Wilkes's classic paper on the use of floating addresses in 1953.

Section (*e*) is the work of several contributors. The first part deals with the basic principles of business data processing, and the facilities required in a computing system for this type of work. The applications described are limited mainly to the accounting field, and to inventory and scheduling work. A chapter on scientific and engineering applications is included, but here the information is spread so thinly that perhaps it might have been better to omit it entirely. The last chapter is devoted to such topics as language translation, game

playing, literature searching, etc. Here too the main value of this chapter is the list of references.

Section (*d*) is also a collection of individual contributions. These deal with basic circuitry, magnetic cores, transistor circuits, logical design, storage, input-output equipment, and reliability. The section on storage is mainly about magnetic drums and magnetic cores (which also have a chapter to themselves), but acoustic delay lines, electrostatic stores, cryogenic films and parametrons are briefly mentioned. Practically all forms of peripheral equipment are discussed including cathode ray tube display devices, and analogue-digital conversion techniques. This last topic is a convenient introduction to Section (*e*) (Analogue Computers), which covers both design and applications.

The last section (Unusual Computer Systems) is mainly of theoretical interest, being devoted to simple Turing type machines.

The book is an up to date work of reference which should be in the library of every Computing Machine Laboratory. R. A. BROOKER

Theoretical Elasticity. By CARY E. PEARSON. Pp. 218. 48s. 1959. (Harvard University Press and Oxford University Press, London)

This book is one of a series of Harvard Monographs in Applied Science designed for research workers in their respective fields. For this reason it is not primarily a book for beginners in the field of elasticity, but it does bring together in a very clear and concise manner all the different modern methods of tackling problems in elasticity from the purely theoretical approach.

The first two chapters deal with the requisite mathematical knowledge, viz. vectors, suffix notation and cartesian tensors. In these two chapters some examples are given in order that the reader may gain some manipulative experience required for the remaining chapters. Chapters 3 and 4 deal in the usual manner with the stress and strain tensors, but it is made more clear to the reader than is usual in most elementary text-books, what the linear theory of elasticity involves and what terms are being neglected. The basic equations of linear elasticity, including interesting integral results, follow in Chapter 5 and here the equations derived are mainly for isotropic materials, although anisotropy is discussed briefly at the end of the chapter. Chapters 6, 7 may be regarded as the most important of the book since in them are brought together the various general methods of solution of the basic equations. These include, methods of potential theory and special functions, function space methods, variational methods, and mention is made of many different authors who have developed these. The three remaining chapters give the basic results of thermo-elasticity, time dependent problems and some elements of the theory of non-linear elasticity, all of which are fields of modern research. One criticism of these chapters 6–9 is that not sufficient specific references are given to the papers and, or books of some of the other authors cited. If such had been done I think the book would have made a very useful reference book as well as one which has presented many modern methods in a uniform notation. R. M. MORRIS

An Introduction to the Geometry of Numbers. By J. W. S. CASSELS. Pp. 344. DM. 69. 1959. (Springer, Berlin)

Minkowski developed a method of obtaining results of interest in the theory of numbers by use of certain geometric ideas, and he gave accounts of his work in his books *Geometrie der Zahlen* (1896) and *Diophantische Approximationen* (1907). Dr. Cassels puts us greatly in his debt by giving an up to date account of the subject in his book *Diophantine Approximation* (1957) and in this new book *Geometry of Numbers*. Minkowski's theory aims at determining when certain regions in n-dimensional space necessarily contain a point, other than the origin, with integral coordinates; his most famous theorem asserts that this is always the case when the region is convex, symmetrical in the origin, and of volume greater than 2^n. His main concern was with the applications of his theory to the theories of Diophantine Approximation, Quadratic Forms and Algebraic Numbers. Since his time the main emphasis of the subject has shifted to the development of the theory itself. To prove that the subject still has important applications, one has merely to instance its use, in conjunction with other powerful tools, by Davenport and others in striking work on Euclid's Algorithm and on indefinite quadratic forms in many variables. To see the way the theory has been developed and enriched, one could not do better than read Dr. Cassels' book.

While some special results are treated in detail, notably Mordell's theorems on binary cubic forms and Davenport's theorems on the product of three linear forms, and while many other special results are quoted, the book is mainly concerned with the general theorems and theories. It develops Mahler's theory of the space of all lattices and of critical lattices, the Minkowski-Hlawka theorem and related results, the theory of successive minima, the theory of automorphic regions and of isolation results, and the theory of packings. It concludes with a short chapter on inhomogeneous problems.

The work is carefully written. It is well motivated, and interesting to read, even if it is not always easy. A reasonable amount of historical material is included.

The reviewer feels that he must issue one warning and make one criticism. In his account of Fejes Tóth's work on the packing of convex domains in the plane, the author replaces one of Fejes Tóth's slightly vague assertions by a rather more precise one (Proposition 1, page 240), which is unfortunately demonstrably false; fortunately a different interpretation of Fejes Tóth's assertion is demonstrably true and sufficient for the required purpose. In the chapter on inhomogeneous problems, the author refers to, but does not even quote, what is, to the reviewer, the most beautiful theorem of this part of the subject. This is the result of Macbeath that can be stated in its 3-dimensional form as follows:* Let K be any convex body, bounded or unbounded, contained in the positive octant, and containing at least one point with integral coordinates, then K contains a point U with

* *Proof.* Take $U=(u, v, w)$ to be the unique point with integral coordinates, for which $u+v\sqrt[3]{2}+w\sqrt[3]{4}$ attains its minimum value in K. Apply the theorem of Minkowski quoted above to the intersection of K with its reflection in U.

integral coordinates, that is reasonably close to the boundary of K, in that the volume of the intersection of K with its reflexion in U is at most 8.

But these are very minor points and do not affect the conclusion that the author has written an excellent account of an interesting subject—a subject moreover where it is still possible (but not as easy as it once was) for an amateur mathematician to make useful contributions without first acquiring any elaborate technique.

C. A. ROGERS

Testing Statistical Hypotheses. By E. L. LEHMANN. Pp. 361. 88s. 1959. (John Wiley and Sons Inc., New York. Chapman and Hall, London)

To paraphrase the words of a famous motion " the spate of elementary statistical textbooks has increased, is increasing, and ought to be diminished ". It is therefore pleasant to report that Professor Lehmann's book is not just another elementary text but a scholarly delineation of much of the mathematical theory which goes to make up what we may loosely call advanced statistical methods. And in contradistinction to most published literature in statistics he has a great deal that is new to say.

It is difficult to summarise this important work, but it may be well to begin by indicating the statistical-mathematical level expected of the reader. The mathematician who knows no statistics whatever, might, with a great deal of hard work and cross reference, be able to read with understanding. It would be advisable for such a person, however, to have had a detailed course which gives him the basic elementary methods. If he has this knowledge then Professor Lehmann is not easy reading but he will be found stimulating and precise in his formulation of many concepts which the statistician slurs. The practising statistician will find the book difficult, unless he has considerable mathematical maturity. In the reviewer's opinion the title of the book is a misnomer since we are really given advanced statistical theory.

It is clear from the outset that the author was a pupil of J. Neyman since the emphasis of the whole book is, fundamentally, the Neyman-Pearson one, although with the advancing years this has had to be widened considerably in scope. The opening chapter indicates to us that statistical method is a means of reaching a decision. We then have such probability theory (set-theory required) as the author considers necessary for his further development, followed by a modern delineation of the Neyman-Pearson procedure including sequential tests.

The two chapters on unbiasedness are clear but require close attention on the part of the reader. Chapter 4 gives a discussion of the concept in relation to simple tests—such for example as the 2×2 table—while Chapter 5 gives the application to normal theory and to the confidence interval. The development is carried on in the chapter on invariance where most powerful invariant tests, unbiasedness and invariance, and invariant confidence sets are all discussed. The book is rounded off

with chapters on the general linear hypothesis and on the minimax principle.

Mathematical statistics is still a rapidly developing subject and there is doubtless much in this book which many would consider ephemeral. Many others would doubtless consider the strict adherence, where possible, to mathematical rigor, stultifying and hampering to new ideas since most developments of importance have been initiated by the heuristic approach. Nevertheless, it is well occasionally to ask ourselves what we are doing and where we are going and Professor Lehmann is most explicit about this.

We are grateful to him for the restatement of old ideas, and for the stimulating résumé of the researches of himself and others. The high regard for his work which is held by statisticians on this side of the Atlantic will be confirmed by this book.

F. N. DAVID

Real Analysis. By EDWARD JAMES MCSHANE and TRUMAN BOTTS. The University Series in Undergraduate Mathematics. Pp. ix, 272. 49s. 6d. 1959. (D. van Nostrand)

This is an admirable account of the abstract parts of real analysis, covering the principles of that theory and offering an introduction to the theory of function spaces, while omitting the detailed discussion of manipulative techniques which the reader is expected to find in conventional textbooks.

After a preliminary section on notations, mainly logical and set theoretic, the book opens with an axiomatic account of the real numbers, which is presented in a way which relates them to the general theory of ordered fields. The interest in partially ordered sets is a notable feature of the treatment, found again in the second chapter which discusses convergence in both topological and in ordered spaces, using for this an abstract technique which we may describe as combining the ideas of filter theory with the language of the theory of directed sets. The third chapter discusses continuity and semicontinuity, and includes an account of the Stone-Weierstrass approximation theorem and of Ascoli's theorem on equicontinuity.

The next two chapters treat the Lebesgue-Stieltjes integral, defining the Riemann integral as a special case. The method used depends greatly on orderings of functions, being a modification of the Daniell procedure. This is followed by a discussion of general measures, the decomposition of measures and the differentiability of Lebesgue integrals. The theorem on substitution in multiple integrals is also proved here.

The last chapter deals with the L^p spaces, treating them by the methods of the theory of linear spaces; it includes an account of Fourier integrals and transforms and of the decomposition of hermitian operators in Hilbert Space.

The chapters are provided with examples illustrating the general theory. The book closes with appendices on transfinite induction, the

maximality principle and the wellordering theorem, and Tychonoff's theorem. There is a bibliography, an index of notations, and a general index.

The production of the book is good, and the text appears to be careful and accurate. The book can be strongly recommended.

J. L. B. COOPER

Ramanujan: Twelve lectures on subjects suggested by his life and work. By G. H. HARDY. (Chelsea Publishing Company, N.Y.)

Since the publication of Ramanujan's Notebooks there has been renewed interest in his work, and this reprint, in slightly reduced format, of Hardy's book of lectures is therefore particularly welcome. The first edition was reviewed very fully in the Gazette (Vol. 148, p. 642) by Professor Mordell, so that a detailed review of the reprint is unnecessary. On rereading the book one is again impressed by the excellence of Hardy's exposition and the elegance of his literary and mathematical style.

R. A. RANKIN

Introduction to the Theory of Groups. By P. S. ALEXANDROFF. Pp. 112. 1959. 17s. 6d. (Blackie)

The theory of groups is so important that it is desirable for a mathematician to learn something of it as early as possible. Professor Alexandroff's little book ably translated by Hazel Perfect and G. M. Petersen, was written so that boys and girls in the top form of a grammar school may study the subject without great difficulty. The abstract ideas are always illustrated by well-chosen examples, many of which will be familiar.

One rather unusual feature is that the group operation is always expressed by $+$, even when it would be more natural to express it by $\times$. The reason for this is explained at the beginning of Chapter III, but it still seems unfamiliar.

This book can be recommended strongly to all sixth-form pupils who intend to study mathematics at a university, and to university students beginning the subject.

E. J. F. PRIMROSE

Brief Reviews February 1961. Errata.

Page 80. The author's name is misspelt in the review of L. M. Milne-Thomson's Theoretical Hydrodynamics, and in the same review the name Plemelj appears as Peomels.

Page 81. For "Funktiaren" read "Funktionen"; for "Craid" read "Braid"; for "Irratraialzahlen" read "Irrationalzahlen".

Page 82. For "in elementary set theory" read "on elementary set theory".

THE LIVERPOOL MATHEMATICAL SOCIETY

Report for the Session 1959–60

12th October 1959. The Society's Prize for 1959 was presented to Mr. T. N. Murphy. Mr. A. W. Fuller, of the Ministry of Education, addressed the Society on "Chords, Cards and Cayley".

17th November 1959. Mr. B. H. P. Rivett, of the National Coal Board, lectured to the Society on "Operational Research in British Industry".

7th December 1959. Dr. A. Hewish of Cavendish Laboratory, Cambridge, gave a lecture on "The Potentialities and Achievements of Large Radio Telescopes".

18th January 1960. The Liverpool Teachers Mathematical Society participated in the Society's discussion of the Ministry of Education Pamphlet "Teaching Mathematics in Secondary Schools" and the Mathematical Association's Report "Mathematics in Secondary Modern Schools". The discussion was opened by Mr. K. Thomas, Mr. C. H. Clarke, and Mr. H. Bailey.

20th February 1960. Members of the Society visited the Liverpool University Computer Laboratory.

7th March 1960. Members Night. The following papers were read:

Mr. G. N. Copley	"Quantity Calculus"
Mr. J. Bunnel	"Some Aspects of Problem Solving"
Dr. F. W. Land	"The OEEC Seminar on New Thinking in School Mathematics"

9th May 1960. The President gave a lecture to the Annual General Meeting of the Society entitled "Round and Around".

The following members were elected to Council for the Session 1960–61: *President*, Dr. G. R. Baldock; *Vice-President*, Miss D. B. Walker; *Treasurer*, Dr. W. F. Newns; *Secretary*, Dr. M. C. R. Butler; *Committee*: Miss A. H. Ainsworth, Mr. J. P. Coles, Mr. P. E. Doak, Dr. T. M. Flett, Miss V. M. Hughes, Mr. K. Thomas, Dr. P. C. Davey.

The Society has about 130 members, and the attendance at meetings averages about 60.

M. C. R. Butler, *Hon. Sec.*

YORKSHIRE BRANCH

Meetings held during Session 1959–1960

17th October 1959. Professor D. N. de G. Allen, of Sheffield University, gave a lecture on "Some aspects of the influence of the needs of modern technology on mathematical teaching and research".

14th November 1959. Mr. S. D. Illingworth and Mr. R. F. Wheeler, introduced a discussion on "Teaching Mathematics in Secondary Schools", Ministry of Education Pamphlet No. 36.

27th February 1960. Dr. J. R. Ravetz, of Leeds University, gave a paper on "The Second Derivative".

7th May 1960. Mr. W. Hope-Jones, formerly of Eton College, addressed the Branch on "Some Fun with Probability".

The average attendance at these meetings has been forty.

During February and March 1960, a course of four lectures was given by Dr. J. Brindley, of Leeds University, on "Mathematics and Meteorology".

Y. J. HAND, *Hon. Sec.*

SOUTHAMPTON AND DISTRICT BRANCH OF THE MATHEMATICAL ASSOCIATION

ANNUAL REPORT FOR 1959–60

The membership of the branch continues to increase, and the average attendance at meetings is now nearly fifty.

The usual series of six meetings commenced on Thursday, 23rd October, 1959, when Mr. and Mrs. A. P. Penfold gave a pair of talks on "Statistics in Schools—past experience and future possibilities". After Mrs. Penfold had introduced the basic methods of statistics, Mr. Penfold told us of the methods by which he introduces statistics to boys throughout his school.

At the annual General Meeting on Thursday, 26th November, 1959, the following committee was elected: *President*, Mrs. W. S. P. Edmunds; *Vice-President*, Professor E. T. Davies; *Secretary and Treasurer*, Dr. F. Rhodes; *University Representative*, Professor B. Thwaites; also Miss M. D. Hemingway, Miss R. A. Clarke, Miss H. Bromby, Mr. P. E. Bryant. The meeting continued with a review of an American report on mathematics given by Dr. W. H. Cockcroft.

A large audience of members and students gathered on Tuesday, 21st January, 1960 to hear Canon D. B. Eperson speak on "Mathematics and Music". He was assisted by Mr. H. L. Haselgrove who produced a multitude of demonstrations to illustrate the lecture, and also by a dozen schoolboys who played musical instruments and manipulated oscilloscopes.

On Friday, 11th March, 1960, we were shown extracts from tele-recordings of programmes on mathematics which have been broadcast by the B.B.C. School Television Series. Mr. D. Gratton, the producer of the programmes explained the aims of the series of programmes and some of the problems encountered in their production.

Professor B. Thwaites lectured on "Models in Mechanics" on Thursday, 5th May, 1960. The last problem which he considered was that of producing a mathematical solution to the medical problem of calculating the velocity of the blood flow from information about the pressure of the blood within the arteries.

At the final meeting of the session on Tuesday, 14th June, 1960, Professor P. B. Morice gave a lecture entitled "Building with Matrices". He showed how engineers calculate the stresses in a statically indeterminate one-dimensional system of rods, and how the systems of equations involved in the calculations can be conveniently reduced to a few matrix equations.

FRANK RHODES, *Hon. Sec.*

EXETER BRANCH OF THE MATHEMATICAL ASSOCIATION

Report for the year 1960

Five meetings have been held during the year and these have been well attended by members, many of whom come considerable distances to reach Exeter—some even from as far afield as Cornwall and North Devon. Details of the meetings are as follows:

19*th February*	Napoleon's Theorem by Mr. A. P. Rollett.
19*th March*	Teaching and examining the VIth forms by Mr. R. V. H. Roseveare.
21*st May*	The Apple and its Sequel by Mr. D. A. Quadling.
18*th November*	Introduction to Groups by Mr. J. Herszberg of Exeter University.

The Summer meeting on 12th June was an expedition to Britannia R.N. College, Dartmouth which was organised by the Mathematical Department of the College. It proved a most interesting and enjoyable venture and the hospitality received was greatly appreciated.

N. A. Comerford, *Hon. Sec.*

MATHEMATICAL ASSOCIATION N.S.W. BRANCH

Report on Activities for 1960

During 1960 six meetings of the N.S.W. Branch of the Mathematical Association were held.

The discussions (begun in 1959) on the Wyndham Report and its implications for the teaching of Mathematics were continued at two meetings in 1960, and the Branch has forwarded the resolutions which emerged from the discussion to the Minister for Education in N.S.W., for his consideration.

Other meetings were held as follows:

6*th May*. Discussion of the 1959 Leaving and Intermediate Certificate Examination papers in Mathematics.

1*st July*. Address by Mr. A. McMullen on the subject " Contemporary Experimental Programmes in High School Mathematics ".

7*th October*. Address by Professor H. O. Lancaster (Professor of Mathematical Statistics at the University of Sydney) on " Some Applications of Statistics to other Sciences ".

4*th November*. Annual General Meeting—Election of office-bearers for 1961; address by the retiring President, Dr. Turner. Prior to the business of this annual meeting, members gathered socially at a dinner.

At 30th September 1960 there were 999 names listed as subscribers to the Journal of the Branch, " The Australian Mathematics Teacher ".

A. R. Bunker, *Hon. Sec.*

NORTH STAFFORDSHIRE AND DISTRICT BRANCH OF THE MATHEMATICAL ASSOCIATION

Report for the year 1960

The Branch now has over 60 members, of which more than half are members of the main Association. During the year five meetings were held at the High School, Newcastle-under-Lyme. These meetings were

well attended by students from local Teachers' Training Colleges and the VIth forms from schools.

17*th February* 1960. Mr. J. C. F. Fair of Alsager Training College gave a talk on Plaited Polyhedra.

5*th April* 1960. Mr. O. F. Phoenix of the North Staffordshire College of Technology gave an illustrated talk on Model Analysis.

27*th September* 1960. A discussion, lead by Mr. C. A. Parrack, of the questions set in the 1960 G.C.E. papers.

26*th October* 1960. Mr. N. G. Dowell gave a talk, illustrated by slides, on the English Electric Data Processing System KDP 10.

30*th November* 1960. The Annual General Meeting was held and was followed by a Mathematical Quiz, organised by Mr. I. R. Vesselo of Alsager Training College. The two teams were drawn from the VIth forms of local grammar schools.

In June the Branch spent a most enjoyable morning at the Computer Section of the English Electric Co. Ltd., at Kidsgrove.

Officers for 1961 have been elected as follows: *President*, Miss D. Ball; *Past President*, Mr. C. A. Parrack; *Chairman*, Mr. F. L. Goodwin; *Treasurer*, Mr. R. B. Hawthorne; *Council Representative*, Mr. I. R. Vesselo; *Minuting Secretary*, Miss S. M. K. Kealey; *Secretary*, Mr. E. R. Kilfoil; *Committee*, Miss D. Grindley, Miss P. Marsden, Mrs. A. P. Williamson, Mr. J. C. F. Fair, Mr. H. W. B. Hayley, Mr. J. M. Todd, Mr. J. Sherratt, Mr. H. Willatt.

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